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SIMPLER PROOFS OF WARING'S THEOREM ON CUBES, WITH VARIOUS GENERALIZATIONS*

BY

L. E. DICKSON

1. **Introduction.** In 1770 Waring conjectured that every positive integer is a sum of nine integral cubes $\geqq 0$. The first proof was given by Wieferich;† but owing to a numerical error he failed to treat a wide range of numbers corresponding to $\nu=4$. Bachmann‡ indicated a long method to fill the gap, but himself made certain errors. The latter were incorporated in the unsuccessful attempt by Lejneek.§ The gap was first filled by Kempner.‖

All of these writers make use of three tables. The computation of each of the last two tables is considerably longer than the first. The third table as given by Wieferich and reproduced by Bachmann contains six errors, corrected by Kamke (cf. Kempner, Mathematische Annalen, loc. cit., p. 399). It is shown here that the last two tables may be completely avoided. The resulting simple proof of Waring's theorem in §§2, 3 is based on the customary prime 5. The second simple proof in §4 is based on the prime 11. By §5, we may also use the prime 17.

However, the main object of the paper is to prove generalizations of two types. Let C_n denote the sum of the cubes of n undetermined integers $\geqq 0$. Waring's theorem states that C_9 represents all positive integers. It is proved in §§ 4, 5 that tx^3+C_8 represents all positive integers if $1\leqq t\leqq 23$, $t\neq 20$, but not if $t>23$. To complete the discussion for $t=20$ would require the extension of von Sterneck's table from 40,000 to 61,500.

It is proved in §6 that $tx^3+2y^3+C_7$ represents all positive integers if $1\leqq t\leqq 34$, $t\neq 10$, 15, 20, 25, 30. Also that $tx^3+3y^3+C_7$ represents all if $1\leqq t\leqq 9$, $t\neq 5$. Various similar theorems are highly probable in view of Lemma 8. More interesting empirical theorems on cubes were announced by the writer in the American Mathematical Monthly for April, 1927, and on biquadrates in the Bulletin of the American Mathematical Society, May-June, 1927.

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* Presented to the Society, April 15, 1927; received by the editors February 16, 1927.

† Mathematische Annalen, vol. 66 (1909), pp. 99–101.

‡ *Niedere Zahlentheorie*, vol. 2, 1910, pp. 477–8.

§ Mathematische Annalen, vol. 70 (1911), pp. 454–6.

‖ *Über das Waringsche Problem und einige Verallgemeinerungen*, Dissertation, Göttingen, 1912. Extract in Mathematische Annalen, vol. 72 (1912), pp. 387–399.

If N is prime to 6, it is shown in §7 that every integer k is represented by $6x^2+6y^2+6z^2+Nw^3$, and that we may take $w\geqq 0$ if $k\geqq 23^3N$. In §8 is discussed the representation of all large integers by ly^3+C_7 when $l\leqq 5$.

The tables and computations in §§ 2–4, 6 and the first part of §5 were kindly checked with great care by Lincoln La Paz.

2. **Three lemmas needed for Waring's theorem.** We prove the following lemmas.

LEMMA 1. *If p is a prime $\equiv 2 \pmod 3$ and if l is an integer not divisible by p, every integer not divisible by p is congruent modulo p^n to a product of a cube by l.*

From the positive integers $\leqq p^n$ we omit the p^{n-1} multiples of p and obtain $\phi=(p-1)p^{n-1}$ numbers $a_1, \cdots, a_\phi$. Each la_i^3 is not divisible by p and hence is congruent to one of the a's modulo p^n. We shall prove that no two of the la_i^3 are congruent. It will then follow that $la_1^3, \cdots, la_\phi^3$ are congruent to $a_1, \cdots, a_\phi$ in some order. Since every integer not divisible by p is congruent to a certain a_j, it will therefore be congruent to a certain la_i^3.

If possible, let $la_i^3\equiv la_k^3 \pmod{p^n}$. Since $a_i\equiv a_kx \pmod{p^n}$ determines an integer x, we have $x^3\equiv 1$. By Euler's theorem, $x^\phi\equiv 1 \pmod{p^n}$. Since ϕ is not divisible by 3, $\phi=3q+r$, $r=1$ or 2. Hence $x^r\equiv 1$, $x\equiv 1$, $a_i\equiv a_k \pmod{p^n}$, contrary to hypothesis.

LEMMA 2. *Let P and e be given integers $\geqq 0$, such that P is of the form $5+48l$. Then every integer $\geqq P^e\cdot 22^3$ can be represented by $P^e\gamma^3+6(x^2+y^2+z^2)$, where γ, x, y, z are integers and $\gamma\geqq 0$.*

It is known that every positive integer not of the form $4^r(8s+7)$ is a sum of three integral squares. Hence this is true of positive integers congruent modulo 16 to one of the following:

$$1,2,3,4,5,6,8,9,10,11,13,14. \tag{1}$$

If n is any integer, we shall prove that

$$n\equiv P^e\gamma^3+6\mu \pmod{96} \tag{2}$$

has integral solutions γ, μ such that $0\leqq\gamma\leqq 22$, and such that μ is one of the numbers (1). Then there is an integer q for which

$$n=P^e\gamma^3+6\mu+96q=P^e\gamma^3+6m, \qquad m=\mu+16q.$$

When $n\geqq P^e\cdot 22^3$, then $n\geqq P^e\gamma^3$, $m\geqq 0$, whence m is a sum of three integral

squares. Thus Lemma 2 will follow if we show that (2) has solutions of the specified type.

We shall first treat the case $e=0$:

(3) $$n \equiv {}^3\gamma + 6\mu \pmod{96}.$$

The method for (3) is such that, by multiplying it by $P, P^2, \cdots$, we can deduce at once the solvability of (2). With this end in view, we omit 3 and 11 from (1) and obtain the numbers

(4) $$1, 2, 4, 5, 6, 8, 9, 10, 13, 14,$$

whose products by 5 (and hence by P) are congruent modulo 16 to the same numbers (4) rearranged.

At the top of the following table we list certain values of γ and below them the least residues modulo 96 of their cubes. The body of the table shows the residue modulo 96 of $\gamma^3+6\mu$ for certain values (4) of μ.

0	1	2	3	4	5	6	7	8	9	10	11	13	14	15	17	18	22
0	1	8	27	64	29	24	55	32	57	40	83	85	56	15	17	72	88
6	7	14	33	70	35												
12	13	20	39	76	41		67		69		95	1		27	29		
24	25	32	51	88	53					64						0	
30	31	38	57	94	59												
36	37	44	63	4	65		91		93		23						
48	49	56	75	16	77	72		80					8				40
54	55	62	81	22	83												
60	61	68	87	28	89		19		21		47						
78	79	86	9	46	11												
84	85	92	15	52	17		43		45		71	73		3	5		

In the body of the table occur $0, 1, \cdots, 95$ with the exception* of

(5) $$2, 10, 18, 26, 34, 42, 50, 58, 66, 74, 82, 90.$$

The latter give all the positive integers <96 of the form $2+8r$.

But 3 and 11 are also available values of μ. For

(6) $$\gamma = 0, 2, 4, 6, 8, 10,$$

the residues modulo 96 of $\gamma^3+6\cdot 3$ and $\gamma^3+6\cdot 11$ are together found to be the numbers (5). This can be proved without computation as follows. In (6), $\gamma=2g$, $g=0, 1, 2, 3, 4, 5$. Thus $\gamma^3+18=2+8(g^3+2)$, $\gamma^3+66=2+8(g^3+8)$. Hence it remains only to show that the values of g^3+2 and

* However large a γ we take, we cannot reach an exceptional number (5). For $\gamma^3+6\mu\equiv 2+8r \pmod{96}$ implies that γ is even and hence $6\mu\equiv 2 \pmod 8$, $\mu\equiv 3 \pmod 4$, $\mu\equiv 3, 7, 11, 15 \pmod{16}$. But none of these four occur in (4).

g^3+8 are together congruent to 0, 1, $\cdots$, 11 modulo 12. But $g^3\equiv g \pmod 6$. Hence g^3+2 takes six values incongruent modulo 6 and therefore also modulo 12. Likewise for g^3+8. But $g^3+2\equiv G^3+8 \pmod{12}$ would imply $g\equiv G \pmod 6$, $g=G$, a contradiction.

Hence for every integer n, (3) has integral solutions γ, μ, $0\leqq\gamma\leqq 22$, μ in (1).

In $5(2+8r)=2+8\rho$, $\rho=1+5r$ ranges with r over a complete set of residues modulo 12. In other words, the products of the numbers (5) by 5 are congruent modulo 96 to the same numbers (5) rearranged. The same is true of their products by $P=5+48l$, since $2k\cdot P\equiv 2k\cdot 5 \pmod{96}$. Evidently the products of 0, 1, $\cdots$, 95 by P are congruent modulo 96 to 0, 1, $\cdots$, 95 rearranged. Hence the products of the numbers in the above table by P are congruent to the same numbers modulo 96. Those numbers are therefore the residues modulo 96 of $P(\gamma^3+6\mu)$ for $0\leqq\gamma\leqq 22$ and for μ in (4). We saw that the products $P\mu$ are congruent modulo 16 to the same numbers (4) rearranged. Hence the residues modulo 96 of $P\gamma^3+6\nu$ for $0\leqq\gamma\leqq 22$ and for ν in (4) are the numbers in the table and hence are the numbers 0, 1, $\cdots$, 95 other than (5).

To complete the proof of the statement concerning (2) when $e=1$, it remains to show that, by choice of γ in (6) and for $t=18$ or 66, $P\gamma^3+t$ is congruent modulo 96 to any assigned number in (5). Since the last was proved for γ^3+t, we need only show that γ^3 and $P\gamma^3$ take the same values modulo 96 when γ takes the values (6). Then $\gamma=2g$, $g=0, 1, 2, 3, 4, 5$. Thus $g^3\equiv 0, 1, 8, 3, 4, 5$; $5g^3\equiv 0, 5, 4, 3, 8, 1 \pmod{12}$, respectively. Hence γ^3 and $5\gamma^3$ take the same values modulo $8\cdot 12$. But the products of 5 and $P=5+48l$ by the same even number γ^3 are congruent modulo 96.

The insertion of the factor P may be repeated e times. This proves the statement concerning (2).

LEMMA 3. *Given the positive numbers s and t and a number B for which $0\leqq B\leqq s$, $t\leqq 9^3 s$, we can find an integer $i\geqq 0$ such that*

$$B \leqq s - ti^3 < B + 3(ts^2)^{1/3}. \tag{7}$$

Denote the last member of (7) by L. If $s<L$, take $i=0$. Next, let $s\geqq L$ and determine a real number r so that $s-tr^3=B$. Then

$$tr^3 = s - B \geqq L - B = 3(ts^2)^{1/3} \geqq t/27, \quad 3r \geqq 1.$$

We may write $r=i+f$, where $0\leqq f<1$, and i is an integer $\geqq 0$. Since $i\leqq r$, $B=s-tr^3\leqq s-ti^3$, as desired in (7). Next,

$$s - ti^3 - B = s - t(r-f)^3 - s + tr^3 = tw,$$

where

$$w = r^3 - (r - f)^3 = 3r^2f - f^2(3r - f) < 3r^2f < 3r^2,$$

since $3r \geqq 1, f < 1$. Since $B \geqq 0$,

$$tr^3 \leqq s, \quad r^2 \leqq (s^2/t^2)^{1/3}, \quad s - ti^3 - B < 3tr^2 \leqq 3(ts^2)^{1/3}.$$

3. **Proof of Waring's theorem.** We first prove that every integer s exceeding $9 \cdot 5^{12}$ is a sum of nine integral cubes $\geqq 0$. For this proof we take $C=9, p=5, t=1$ in our formulas. Since $s > Cp^{3\cdot 4}$ there exists an integer $n \geqq 4$ such that

$$Cp^{3n} < s \leqq Cp^{3(n+1)}. \tag{8}$$

Write

$$k = 3(tC^2)^{1/3}p^{2n+2}. \tag{9}$$

Hence

$$3(ts^2)^{1/3} \leqq k. \tag{10}$$

We separate two cases. First, let $Cp^{3n}+2k \leqq s$. Then Cp^{3n} and $Cp^{3n}+k$ are both $\leqq s$. Taking them in turn for B in Lemma 3, and using (10), we conclude that there exist integers I and J, each $\geqq 0$, such that

$$Cp^{3n} \leqq s - tI^3 < Cp^{3n} + k,$$
$$Cp^{3n} + k \leqq s - tJ^3 < Cp^{3n} + 2k.$$

Hence there are two distinct integral values I and J of i which satisfy

$$Cp^{3n} \leqq s - ti^3 < Cp^{3n} + 2k, \quad i \geqq 0. \tag{11}$$

Second, let $Cp^{3n}+2k > s$. Then (11) holds for $i=0$ and (when $t=1$) for $i=1$, since the integer Cp^{3n} is less than s and hence is $\leqq s-1$.

Hence in both cases there exist two distinct integers and hence two consecutive integers $j-1$ and j, which are both values of i satisfying (11).

At least one of the integers $s-t(j-1)^3$ and $s-tj^3$ is not divisible by 5. For, their difference is the product of t by $3j^2-3j+1$. The double of the latter is congruent to $(j+2)^2-2$, modulo 5. But 2 is not congruent to a square.

Hence there exists an integer $a \geqq 0$ such that (11) holds when $i=a$, and such that $s-ta^3$ is not divisible by $p=5$. By Lemma 1, there exist integers b and M such that

$$s - ta^3 = b^3 + p^nM, \quad 0 < b < p^n. \tag{12}$$

When $n \geqq 4$, we have

$$Cp^{3n} + 2k \leqq 12p^{3n}. \tag{13}$$

For, if we insert the value (9) of k, divide all terms by p^{3n}, and note that $1/p^{n-2} \leqq 1/p^2$, we see that (13) holds if

$$C + \frac{6}{p^2}(tC^2)^{1/3} \leqq 12. \tag{14}$$

When $C=9$, $p=5$, this holds if

$$t \leqq \frac{25^3}{8 \cdot 9^2} = 24.1.$$

By (11) with $i=a$, (12) and (13), we get

$$Cp^{3n} \leqq b^3 + p^n M < 12p^{3n}, \quad (C-1)p^{3n} < Cp^{3n} - b^3.$$

Hence

$$(C-1)p^{2n} < M < 12p^{2n}.$$

Write $M = N + 6p^{2n}$. Thus

$$(C-7)p^{2n} < N < 6p^{2n}, \tag{15}$$

$$s = ta^3 + b^3 + p^n(N + 6p^{2n}). \tag{16}$$

We seek integers c and m, each $\geqq 0$, such that

$$p^n N = c^3 + p^n \cdot 6m, \quad m = d_1^2 + d_2^2 + d_3^2, \tag{17}$$

for integers d_i. Then will

$$s = ta^3 + b^3 + c^3 + p^n(6p^{2n} + 6m). \tag{18}$$

Writing A for p^n, we then have

$$s = ta^3 + b^3 + c^3 + \sum_{i=1}^{3} [(A + d_i)^3 + (A - d_i)^3]. \tag{19}$$

These cubes are all $\geqq 0$. For, if $d_i^2 > A^2$, then $m > A^2 = p^{2n}$, and, by (17), $p^n N > 6p^n p^{2n}$, contrary to (15). Hence s is a sum of nine integral cubes $\geqq 0$.

It remains to select c and m. Choose an integer e so that

$$e = 0, 1, 2, \quad e + n \equiv 0 \pmod 3. \tag{20}$$

The condition in Lemma 2 is $N \geqq 5^e \cdot 22^3$. By (15), this will be satisfied if $(C-7)p^{2n} \geqq 5^e \cdot 22^3$. When $n \geqq 4$, the minimum value of $2n - e$ is 6. Hence it suffices to take

$$(21)\qquad (C-7)5^6 \geqq 22^3, \quad C-7 \geqq \left(\frac{22}{25}\right)^3 = (0.88)^3 = 0.681472.$$

Thus if $C \geqq 7.682$, Lemma 2 shows the existence of integers γ and m, each $\geqq 0$, such that $N = 5^e\gamma^3 + 6m$, where m is a sum of three integral squares. By (20), $5^{e+n}\gamma^3$ is the cube of an integer $c \geqq 0$. Thus (17) holds when $p=5$.

This completes the proof that every integer s exceeding $9 \cdot 5^{12}$ is a sum of nine integral cubes $\geqq 0$. The same is true when $s<40{,}000$ by the table of von Sterneck,* which shows also that if $8042<s<40{,}000$, s is a sum of six integral cubes $\geqq 0$. To utilize the latter result, let $10^4 \leqq s \leqq 9 \cdot 5^{12}$. By Lemma 3 with $B=10^4$, there exists an integer $u \geqq 0$ satisfying

$$(22)\qquad 10^4 \leqq \sigma < 10^4 + 3(ts^2)^{1/3}, \qquad \sigma = s - tu^3.$$

We have $s<5^{14}$. For $t<5^2$, the radical is $<5^{10}$. Also, $10^4 = 5^4 2^4 < 5^9$. Hence $\sigma < 16 \cdot 5^9 < 4^3 \cdot 5^9$.

Apply Lemma 3 with $t=1$, $B=10^4$, and s replaced by σ. Thus there exists an integer $v \geqq 0$ satisfying

$$(23)\qquad 10^4 \leqq \tau < 10^4 + 3\sigma^{2/3}, \qquad \tau = \sigma - v^3.$$

The radical is $<4^2 \cdot 5^6$. Also, $10^4 < 4^2 \cdot 5^6$. Hence $\tau < 4^3 \cdot 5^6$. As before, there exists an integer $w \geqq 0$ satisfying

$$(24)\qquad 10^4 \leqq \tau - w^3 < 10^4 + 3\tau^{2/3} = 4 \cdot 10^4 = 40{,}000.$$

Since $\tau - w^3$ is therefore a sum of six cubes, while $s = tu^3 + v^3 + \tau$, s is a sum of nine integral cubes $\geqq 0$. This completes the proof of Waring's theorem.

4. **The first generalizations.** Let C_n denote the sum of the cubes of n undetermined integers $\geqq 0$. Let t be an integer $\geqq 0$.

LEMMA 4. *The form $f_t = tx^3 + C_8$ represents all positive integers $\leqq 40{,}000$ if and only if $0<t\leqq 23$.*

If $t>23$ or if $t=0$, C_8 and hence f_t fail to represent 23. Next, let $0<t\leqq 23$. By von Sterneck's table, every positive integer $\leqq 40{,}000$, except 23 and 239, is a sum of eight integral cubes $\geqq 0$. It remains only to show that f_t represents 23 and 239. Take $x=1$. Since

$$0 \leqq 23 - t < 23, \quad 23 < 239 - t < 239,$$

both $23-t$ and $239-t$ are represented by C_8.

* Akademie der Wissenschaften, Wien, Sitzungsberichte, vol. 112, IIa (1903), pp. 1627–1666. Dahse's table to 12,000 was published by Jacobi, Journal für Mathematik, vol. 42 (1851), p. 41; Werke, vol. 6, p. 323.

THEOREM I. *If* $1 \leqq t \leqq 23$, $t \neq 20$, *every positive integer is represented by* $f_t = tx^3 + C_8$.

We proceed as in §3 with $p=5$ or $p=11$ according as t is not divisible by 5 or 11, and with $n \geqq 4$ or $n \geqq 3$, respectively. We shall find limits within which C may be chosen. But we refrain from making a definite choice for C initially, since we may need to decrease C slightly to meet the difficulty arising below (11) when $t>1$. Then (11) does not hold for $i=1$ if

$$Cp^{3n} > s - t. \tag{25}$$

In the latter case, we employ a new constant C'. Then

$$C'p^{3n+3} = Cp^{3n} \cdot p^3C'/C > (s-t)p^3C'/C$$

will be $\geqq s$ if

$$C' \geqq \frac{C}{p^3} \cdot \frac{s}{s-t},$$

and hence if $C' > \frac{1}{2}C$. Thus if C' lies between $\frac{1}{2}C$ and C, (8) will remain true after C is replaced by C'. By (25), $Cp^{3n} = s-t+P$, $P>0$. By (8), $P<t\leqq 23$. Write $q=P/p^{3n}$. Since $n\geqq 4$ or $\geqq 3$, according as $p=5$ or 11, q is very small. We take $C'=C-q$. Then C' lies between $\frac{1}{2}C$ and C, and $C'p^{3n}=s-t$. Hence after taking C' as a new C, we have (8) and the desired two integral solutions i of (11) in all cases.

For $p=5$, $t\leqq 23$, (14) holds when $C\leqq 9.03$. Reduction to $C=9$ permits us to avoid the difficulty mentioned before. The entire proof in §3 now holds if $p=5$ and if t is not divisible by 5.

LEMMA 5. *Let* $P=11+48l$ *and* e *be given integers* $\geqq 0$. *Every integer* $\geqq P^e \cdot 23^3$ *is represented by* $P^e\gamma^3+6(x^2+y^2+z^2)$, $\gamma\geqq 0$.

We now omit 4, 5, and 13 from the available numbers (1) and have

$$1,2,3,6,8,9,10,11,14, \tag{26}$$

whose products by P are congruent modulo 16 to the same numbers (26) rearranged.

The following table shows the residues modulo 96 of $\gamma^3+6\mu$ for $\gamma=0, 1, \cdots, 23$ and for μ in (26). It was computed as in §2, with also $19^3\equiv 43$, $21^3\equiv 45$, $23^3\equiv 71$ (mod 96).

$\gamma=0$	1	2	3	4	5	6	7	8	9	10	11
6	7	14	33	70	35	30		38		46	
12	13	20	39	76	41				69		
18	19	26	45	82	47	42	73	50		58	5
36	37	44	63	4	65		91				23
48	49	56	75	16	77	72		80	9	88	
54	55	62	81	22	83	78		86		94	
60	61	68	87	28	89				21		
66	67	74	93	34	95	90	25	2	27	10	53
84	85	92	15	52	17		43				71

6μ	$\gamma=13$	14	15	17	18	19	21	22	23
12	1			29			57		
36			51			79			11
48		8			24			40	
84			3			31			59

In the body of the table occur $0, 1, \cdots, 95$ with the exception of

$$0, 32, 64. \tag{27}$$

Since $32P\equiv 64$, $64P\equiv 32$ (mod 96), and since the products of $0, 1, \cdots, 95$ by P are evidently congruent modulo 96 to the same numbers rearranged, the same is true of the numbers in the table. Using the omitted value 4 of μ, we get

$$18^3+24\equiv 0,\quad 2^3+24\equiv 32,\quad 10^3+24\equiv 64 \pmod{96}. \tag{28}$$

Since $4(P^2-1)\equiv 4(11^2-1)\equiv 0 \pmod{96}$, $4P^{2k}\equiv 4$, and multiplication of (28) by P^{2k} yields

$$18^3P^{2k}+24\equiv 0,\quad 2^3P^{2k}+24\equiv 32,\quad 10^3P^{2k}+24\equiv 64 \pmod{96}. \tag{29}$$

This completes the proof of Lemma 5 where e is even.

Since $P+1$ is divisible by 12, the product of an even cube by P is congruent to its negative, modulo 96. Hence

$$6^3P\equiv -24,\quad 22^3P\equiv 8,\quad 14^3P\equiv 40 \pmod{96}. \tag{30}$$

As before, multiplication of (30) by P^{2k} yields

$$6^3P^e+24\equiv 0,\quad 22^3P^e+24\equiv 32,\quad 14^3P^e+24\equiv 64 \pmod{96}, \tag{31}$$

where $e=2k+1$. Thus Lemma 5 follows when e is odd.

Let $p=11$, $n\geqq 3$, $t\leqq 15$, $t\neq 11$. Then (13) holds if

$$C+\frac{6}{11}(tC^2)^{1/3}\leqq 12 \qquad \text{for } t=15. \tag{32}$$

When $C=7.05$, the left member is 11.9960. By (15), the condition in Lemma 5 is satisfied if $(C-7)11^{2n}\geqq 11^e\cdot 23^3$. By (20), the minimum of $2n-e$ for $n\geqq 3$ is 6. Hence it suffices to take

$$(33)\qquad (C-7)11^6 \geqq 23^3, \qquad C-7 \geqq 0.006868.$$

Hence all the conditions on C are satisfied if $C=7.01$, and the reduction from 7.05 avoids the difficulty arising when (25) holds. Next,

$$4(3j^2-3j+1) \equiv (j+5)^2+1 \pmod{11},$$

while -1 is not congruent to a square. For $C=7.01$, the proof in §3 now shows that every integer s exceeding $C\cdot 11^9$ is represented by f_t. It remains to prove this also when $10^4\leqq s\leqq C\cdot 11^9$. Consider (22) and (23). Now

$$(34)\qquad (tC^2)^{1/3} = 9.033, \quad 10^4 < (0.01)11^6, \quad \sigma < (27.11)11^6,$$

$$\sigma^{2/3} < (9.0245)11^4, \quad 10^4 < (0.69)11^4, \quad \tau < 28\cdot 11^4 < 7^3\cdot 11^3.$$

In place of (24), we now have $10^4\leqq \tau-w^3<28{,}000$. This proves Theorem I when $t\leqq 15$, $t\neq 11$.

5. **The case $t=20$.** First, take $p=11$. The proof fails if $n=3$, since (32) requires $C<7$. Hence $n\geqq 4$, and (14) holds if $C\leqq 11.3$. Thus every s exceeding $C\cdot 11^{12}$ is represented by f_{20}, where $C^2=50$. But if $s<C\cdot 11^{12}$, we obtain by (22)–(24) the condition $\tau-w^3<152{,}794$, which is far beyond the limit of von Sterneck's table.

A better result is obtained by taking $p=17$, $n\geqq 3$. Lemma 2 holds also when $P=17+48l$. For, by multiplying (3) by this P, we see that every integer is congruent modulo 96 to $P\gamma^3+6P\mu$. But $6P\equiv 6 \pmod{96}$. If $3j^2-3j+1\equiv 0 \pmod{17}$, multiplication by 6 gives $(j+8)^2\equiv 7$, whereas 7 is a quadratic non-residue of 17. The two conditions on C are both satisfied if $C^2=50$. Then (22)-(24) yield $\tau-w^3<65{,}500$. A still lower limit will be found by employing

LEMMA 6. *Given the positive numbers s and t, and a number B for which $0\leqq B\leqq s$, $t\leqq 9^3 s$, we can find an integer $i\geqq 0$ such that*

$$B \leqq s - ti^3 < B + t(3r^2-3r+1), \quad r^3 = (s-B)/t.$$

The proof consists in the following modification of the last part of the proof of Lemma 3. The condition for $w<3r^2-3r+1$ is

$$(1-f)[3r^2-3r(1+f)+1+f+f^2] > 0.$$

Since $0\leqq f<1$, this is evidently satisfied when $r\geqq 1+f$. In the contrary case, $i=0$, $r=f$, and the quantity in brackets is $(1-f)^2>0$.

By (21) with 5 replaced by 17, $C-7\geqq 0.0004411$. This and condition (13) are both satisfied if $C=7.00045$. It remains to treat integers $s<C\cdot 17^9$. By Lemma 4 we may take $s>40{,}000$. By Lemma 6 with $t=20$, $B=8043$, there exists an integer $u\geqq 0$ such that

$$8043 \leqq \sigma < 8043 + 20(3r^2 - 3r + 1), \quad \sigma = s - 20u^3,$$

where $r^3=s/20$ slightly exceeds the initial r^3. Then

$$\log r = 3.5393787, \quad r^2 = 11{,}988{,}290, \quad r = 3462, \quad \sigma - 8043 < 719{,}089{,}700.$$

Apply Lemma 6 with $t=1$, $B=8043$, s replaced by σ. Hence there exists an integer $v\geqq 0$ such that

$$8043 \leqq \tau < 8043 + 3R^2 - 3R + 1, \quad \tau = \sigma - v^3, \quad R^3 = \sigma - 8043,$$

$$\log R = 2.9522610, \quad R^2 = 802{,}642.2, \quad R = 895.9, \quad \tau - 8043 < 2{,}405{,}240.$$

By Lemma 6 with $t=1$, there exists an integer $w\geqq 0$ such that

$$8043 \leqq \tau - w^3 < 8043 + 3\rho^2 - 3\rho + 1, \quad \rho^3 = \tau - 8043,$$

$$\log \rho = 2.1270528, \quad \rho^2 = 17{,}951.7, \quad \rho = 134, \quad \tau - w^3 < 61{,}497.$$

Hence Theorem I would hold also for $t=20$ provided an extension of von Sterneck's table would show that every integer between 8043 and 61,497 is a sum of six cubes.

To prove Waring's theorem by means of $p=17$, $n\geqq 3$, and the same C, we find by three applications of Lemma 3 with $t=1$, $B=8043$, that $\tau-w^3<42{,}846.7$. This limit is reduced by using Lemma 6.

6. **The second generalizations.** We employ two lemmas.

LEMMA 7. *$F_l=ly^3+C_7$ represents all positive integers $\leqq 40{,}000$ if and only if $l=2$-6, 9-15. F_7 represents all $\leqq 40{,}000$ except 22. F_8 represents all except 23, 239, and 428.*

By the tables of Dahse and von Sterneck, C_7 represents every positive integer $\leqq 40{,}000$ except

(35) 15, 22, 23, 50, 114, 167, 175, 186, 212, 231, 238, 239, 303, 364, 420, 428, 454.

Thus $F_0\neq 15$. Also $F_1=C_8\neq 23$. If $l>15$, evidently $F_l\neq 15$. Hence let $2\leqq l\leqq 15$. The successive differences of the numbers (35) are

(36) 7, 1, 27, 64, 53, 8, 11, 26, 19, 7, 1, 64, 61, 56, 8, 26.

Hence every positive difference of two numbers (35), not necessarily consecutive, is 1, 7, 8, 11, or is >15.

First, let $l\neq 7, 8, 11$. If n and m $(n>m)$ are any two numbers (35), then $n-m\neq l$. Since $n-l$ is therefore not one of the numbers (35), it is represented by C_7. Hence n is represented by F_l with $y=1$.

A like result holds also if $l=11$. By (36) the only pair of numbers (35) with the difference 11 is the pair 186, 175. But $186-11\cdot 2^3=98$ is not in (35) and hence is represented by C_7. Hence $F_{11}=186$ for $y=2$.

For $l=7$, it remains to consider $n=22$ and 238, which alone exceed predecessors by 7, as seen from (36). But $238-7\cdot 2^3=182$ is not in (35) and hence is represented by C_7.

Finally, for $l=8$, (36) shows that only $n=23$, 175, 239, and 428 exceed smaller numbers in (35) by 8. Since F_8 is a sum of eight cubes, it does not represent 23 or 239. Next, $175-8\cdot 2^3=111$ is not in (35). But $428-8=420$, $428-8\cdot 2^3=364$, and $428-8\cdot 3^3=212$ are all in (35), while $428<8\cdot 4^3$. Hence $F_8\neq 428$.

Lemma 8. *$F_{k,l}=kx^3+ly^3+C_7$ represents all positive integers $\leqq 40{,}000$ when $l=2$–6, 9–15, and k is arbitrary; when $l=7$ if and only if $1\leqq k\leqq 22$; when $l=8$ if and only if $1\leqq k\leqq 23$; but not if both k and l exceed* 15.

In the final case, $F\neq 15$. The first case follows from Lemma 7. Next, let $l=7$. If $k=1$, F is $7y^3+C_8$, which represents all integers $\leqq 40{,}000$ by Lemma 4. If $k>22$, $F=22$ requires $x=0$, whereas $7y^3+C_7\neq 22$ by Lemma 7. It remains to consider the case $l=7$, $1<k\leqq 22$. By Lemma 7, we have only to verify that $F=22$ has integral solutions. When $k=7$, take $x=y=1$, since C_7 represents 8. When $k\neq 7$, take $x=1$, $y=0$, since C_7 represents $22-k$, which is $\geqq 0$, <22, and $\neq 15$.

Finally, let $l=8$. If $k=1$, apply Lemma 4. If $k>23$, $F=23$ implies $x=0$, whereas $8y^3+C_7\neq 23$ by Lemma 7. Hence let $1<k\leqq 23$. By Lemma 7, we have only to verify that F represents 23, 239, 428. If $k\neq 8$, take $x=1$, $y=0$; then $F=k+C_7$ represents 23, since C_7 represents $23-k\neq 15$, 22, 23; $F=239$, since $239-k$ is not 231 and is in the interval from 216 to 237 and hence is represented by C_7; $F=428$, since $428-k$ is not 420 and is in the interval from 405 to 426 and hence is represented by C_7. If $k=8$, take $x=y=1$ and note that C_7 represents 7, 223, and 413.

Theorem II. *$tx^3+ly^3+C_7$ represents all positive integers if $l=2$, $1\leqq t\leqq 34$, $t\neq 10$, 15, 20, 25, 30, and if $l=3$, $1\leqq t\leqq 9$, $t\neq 5$.*

Let neither t nor l be divisible by the prime $p\equiv 2 \pmod 3$. By §§ 3, 4, there exists an integer $a\geqq 0$ such that (11) holds when $i=a$ and such that $s-ta^3$ is not divisible by p. By Lemma 1, there exist integers b and M such that

$$(37) \qquad s - ta^3 = lb^3 + p^n M, \qquad 0 < b < p^n.$$

We shall presently choose C and t so that (13) is satisfied. Using also (11) with $i = a$, we have

$$Cp^{3n} \leqq lb^3 + p^n M < 12p^{3n}, \quad (C - l)p^{3n} < Cp^{3n} - lb^3.$$

Hence

$$(C - l)p^{2n} < M < 12p^{2n}.$$

Write $M = N + 6p^{2n}$. Then

$$(38) \qquad (C - l - 6)p^{2n} < N < 6p^{2n}, \quad s = ta^3 + lb^3 + p^n(N + 6p^{2n}).$$

(I) Let $p = 5$. As in (21), the condition $N \geqq 5^6 \cdot 22^3$ in Lemma 2 is satisfied if $C - l - 6 \geqq 0.68148$. Since $t \geqq 1$, (13) fails if $n = 3$. Hence $n \geqq 4$.

First, let $l = 2$ and take $C = 8.68148$. Condition (14) gives $t \leqq 35.076$. Hence 34 is the maximum t. It remains to consider integers s satisfying $10^4 \leqq s \leqq C \cdot 5^{12}$. Since $t < 5^3$, $C^2 < 5^3$, the radical in (22) is $< 5^{10}$, and $\sigma < 16 \cdot 5^9$. By Lemma 3 with $B = 10^4$, $t = 2$, and s replaced by σ, there exists an integer $v \geqq 0$ such that

$$(39) \qquad 10^4 \leqq \tau < 10^4 + 3(2\sigma^2)^{1/3}, \qquad \tau = \sigma - 2v^3.$$

Since $2\sigma^2 < 4^6 \cdot 5^{18}$, we have (24). This proves Theorem II for $l = 2$, $t \leqq 34$, t prime to 5.

Second, let $l = 3$ and take $C = 9.68148$. By (14), $t \leqq 9.6186$. Hence $t \leqq 9$. Since $tC^2 < 10^3$, (22) gives $\sigma < 31 \cdot 5^8 < 7 \cdot 5^9$. By Lemma 3 with $B = 10^4$, $t = 3$, and s replaced by σ,

$$10^4 \leqq \tau < 10^4 + 3(3\sigma^2)^{1/3}, \qquad \tau = \sigma - 3v^3.$$

Since $3\sigma^2 < 4^6 \cdot 5^{18}$, (24) holds. This proves Theorem II for $l = 3$, $t \leqq 9$, $t \neq 5$.

Finally, if $l \geqq 4$, then $C \geqq 10.682$, and (14) fails if $t \geqq 2$. But if $t = 1$, we have the form treated in §4.

(II) Let $p = 11$. Whether $n \geqq 3$ or $n \geqq 4$, the condition in Lemma 5 is satisfied if $C - l - 6 \geqq 0.006868$, as in (33).

First, let $n \geqq 3$. If $l \geqq 3$ and $t \geqq 5$, (32) fails. Hence let $l = 2$, $C = 8.006868$. Then $C^2 = 64.11$ and (32) requires that $t \leqq 6$. But (32) holds if $t = 5$ since $(5C^2)^{1/3} < 6.844$. The only new case is $t = 5$. It remains to consider integers s satisfying $10^4 \leqq s \leqq C \cdot 11^9$. We employ (22), (39), and (24):

$$10^4 < (0.006)11^6, \quad \sigma < (20.538)11^6, \quad (2\sigma^2)^{1/3} < (9.45)11^4,$$

$$\tau < 30 \cdot 11^4 = 330 \cdot 11^3 < 7^3 \cdot 11^3, \quad \tau^{2/3} < 6000, \quad \tau - w^3 < 28{,}000.$$

This proves Theorem II for $t = 5$, $l = 2$.

Second, let $n \geqq 4$, $l=2$. Using the same C, we find that (14) holds when $t \leqq 8145$. But the proof of Theorem II fails for the first new case $t=10$ when $s<C \cdot 11^{12}$. We employ (22), (39), and (24) with the refinement of replacing 10^4 by 8042. We obtain

$$\sigma < (25.8682)11^8, \quad \tau < (73.576)11^5, \quad \tau - w^3 < 163{,}969,$$

where the final number is beyond the limit 40,000 of von Sterneck's table.

7. **Generalization of Lemmas 2 and 5.** These lemmas can be generalized as follows.

THEOREM III. *If N is a positive integer divisible by neither 2 nor 3, every integer* $\geqq 23^3N$ is represented by $N\gamma^3+6(x^2+y^2+z^2)$, where γ, x, y, z are integers and $\gamma \geqq 0$.*

As in the proof of Lemma 2 this will follow from

LEMMA 9. *Every integer n is congruent modulo 96 to $N\gamma^3+6\mu$ for $0 \leqq \gamma \leqq 23$, with μ in the set* (1).

Proof was given in §5 when $N \equiv 17 \pmod{48}$. It is true by the proof in Lemma 2 when $N \equiv 5, 5^2, 5^3 \equiv 29, 5^4 \equiv 1 \pmod{48}$.

If $N=41+48l$, $N \equiv 5^2 \pmod{16}$. We saw that the products of the numbers (4) by 5 are congruent modulo 16 to the same numbers (4) rearranged. Hence the same is true of their products by N. Also $3N \equiv 3 \cdot 9 \equiv 11$, $11N \equiv 3 \pmod{16}$. Hence the products of all the numbers (1) by N are congruent modulo 16 to the same numbers (1) rearranged. Multiplication of (3) by N proves Lemma 9.

For $N=37+48l$, we proceed as in the last part of the proof of Lemma 2. In $37(2+8r)=2+8\rho$, $\rho=9+37r$ ranges with r over a complete set of residues modulo 12. Finally, $37g^3 \equiv g^3 \pmod{12}$. The lemma follows also for $N \equiv 37^3 \equiv 13 \pmod{48}$.

By Lemma 5, the lemma holds when $N \equiv 11$ or $11^3 \equiv 35 \pmod{48}$.

Let $N=19+48l$. The products of the numbers (26) by 3 and hence by N are congruent modulo 16 to the same numbers rearranged. Since $N \equiv 1 \pmod 3$, $32N \equiv 32$, $64N \equiv 64 \pmod{96}$. Since $N+5$ is divisible by 12, the product of an even cube by N is congruent to its product by -5 modulo 96. Hence

$$N \cdot 6^3 \equiv -5 \cdot 24 \equiv -24, \quad N \cdot 22^3 \equiv (-5)(-8) = 40,$$
$$N \cdot 14^3 \equiv (-5)(-40) \equiv 8 \pmod{96}.$$

* Except for $N \equiv 11, 19, 35, 43 \pmod{48}$, we may replace 23 by 22. But when $N=1$, $S=9832$ is between 21^3 and 22^3 and is not represented by $\gamma^3+6(x^2+y^2+z^2)$. For, that requires $\gamma^3 \equiv S \equiv 4$, $\gamma \equiv 4 \pmod 6$. But no one of $(1/6)(S-4^3)=4 \cdot 407$, $(1/6)(S-10^3)=16 \cdot 92$, $(1/6)(S-16^3)=4 \cdot 239$ is a sum of three squares.

Adding 24 to each, we get 0, 64, 32, respectively. The lemma follows also for $N\equiv 19^3\equiv 43 \pmod{48}$.

Let $N=23+48l$. Then $N\equiv 7 \pmod{16}$. Omitting 1, 4, 9 from (1), we get

$$2,3,5,6,8,10,11,13,14, \tag{40}$$

whose products by 7 are congruent modulo 16 to the same numbers permuted. For μ in (40), the residues modulo 96 of $\gamma^3+6\mu$ are shown in the following table having the values of γ at the top:

0	1	2	3	4	5	6	7	8	9	10	11	13	14	15	17	18	22
12	13	20	39	76	41				69			1			29		
18	19	26	45	82	47	42	73	50		58	5			33			
30	31	38	57	94	59	54		62		70							
36	37	44	63	4	65		91				23			51			
48	49	56	75	16	77	72	7	80		88	35		8			24	40
60	61	68	87	28	89				21								
66	67	74	93	34	95	90	25	2	27	10	53	55		81	83		
78	79	86	9	46	11	6		14		22							
84	85	92	15	52	17		43				71			3			

In the body of the table occur $0, 1, \cdots, 95$ with the exception of 0, 32, 64. But $32N\equiv 64$, $64N\equiv 32 \pmod{96}$. We proceed as in the proof of Lemma 5. Since $N+1$ is divisible by 12, the product of an even cube by N is congruent to its negative modulo 96. Hence

$$6^3N\equiv -24,\quad 22^3N\equiv 8,\quad 14^3N\equiv 40 \qquad \pmod{96}.$$

Adding 24, we get 0, 32, 64, respectively. The same proof holds for $N=47+48l$.

For $N\equiv 7$ or $31 \pmod{48}$, the preceding proof is to be modified as for $N=19+48l$.

This completes the proof of Theorem III.

If $0<n<23^3N$ in Lemma 9, write $\Gamma=\gamma-96$. Then

$$n\equiv N\Gamma^3+6\mu \pmod{96},\quad n\geqq N\Gamma^3. \tag{41}$$

If n is negative, write $\Gamma=\gamma-96w$, and choose a positive integer w so that $n\geqq N\Gamma^3$. If $n>23^3N$, take $\Gamma=\gamma$. In every case, (41) holds. As in the proof of Lemma 2, this implies

THEOREM IV. *If N is any integer prime to 6, every integer is represented by $N\Gamma^3+6(x^2+y^2+z^2)$, where the integer Γ may be negative.*

8. **Representation of all large numbers.** We prove the following theorem.

THEOREM V. *For $l=1, 2, 3, 4$, or 5, $F_l=ly^3+C_7$ represents all sufficiently large integers.**

Let r be the real ninth root of $12/(6.9+l)$. Then $r>1$. The number of primes $\equiv 2 \pmod 3$ which exceed x and are $\leqq rx$ is known to increase indefinitely with x. Choose as x the first radical in (42). Hence for all sufficiently large integers n, there exist at least ten primes p such that

$$\left(\frac{n}{12}\right)^{1/9} < p \leqq \left(\frac{n}{6.9+l}\right)^{1/9}, \quad p \equiv 2 \pmod 3. \tag{42}$$

The product of the ten primes exceeds $(n/12)^{10/9}$ and hence exceeds n if $n>12^{10}$. Hence not all ten are divisors of n. Henceforth, let p be a prime $>l$ not dividing n and satisfying (42). By Lemma 1 there exist integers δ and M satisfying

$$n \equiv l\delta^3 \pmod{p^3}, \quad n - l\delta^3 = p^3M, \quad 0 < \delta < p^3.$$

By (42), $(6.9+l)p^9 \leqq n < 12p^9$. Hence

$$(6.9 + l)p^9 - lp^9 \leqq n - l\delta^3 = p^3M, \quad p^3M < n < 12p^9.$$

Cancellation of factors p^3 gives

$$6.9p^6 < M < 12p^6.$$

Write $M=N+6p^6$. Then $0.9p^6<N<6p^6$. Let $p\geqq 11$. Then $N>22^3$. By Lemma 2 with $e=0$, N can be represented by $\gamma^3+6(d_1^2+d_2^2+d_3^2)$ with $\gamma \geqq 0$. If any $|d_i|\geqq p^3$, then $N\geqq 6p^6$, contrary to the above. Hence in

$$\begin{aligned} n &= l\delta^3 + p^3M = l\delta^3 + 6p^9 + p^3\gamma^3 + 6p^3(d_1^2 + d_2^2 + d_3^2) \\ &= l\delta^3 + (p\gamma)^3 + \sum_{i=1}^{3} [(p^3 + d_i)^3 + (p^3 - d_i)^3], \end{aligned}$$

each cube is $\geqq 0$. This proves Theorem V.

The following second proof applies to numbers exceeding a much smaller limit. For n sufficiently large, there exist seven primes P satisfying

$$(n/12)^{1/6} < P \leqq (n/C)^{1/6}, \quad P \equiv 2 \pmod 3, \quad C < 12. \tag{43}$$

The earlier discussion applies when p^3 is replaced by P^2 and gives

$$n = l\delta^3 + P^2M, \quad M = N + 6P^4, \quad (C - l - 6)P^4 < N < 6P^4.$$

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* For $l=1$, the case of 8 cubes, see Landau, Mathematische Annalen, vol. 66 (1909), pp. 102–5; *Verteilung der Primzahlen*, vol. 1, 1909, pp. 555–9. For $l=2$, Dickson, Bulletin of the American Mathematical Society, vol. 33 (1927), p. 299.

Thus $N \geqq 23^3 P$ if

$$l+6+\left(\frac{23}{P}\right)^3 \leqq C < 12, \tag{44}$$

which holds if $C=l+6.9$. Then by Theorem III, N is represented by $P\gamma^3+6(d_1^2+d_2^2+d_3^2)$ with $\gamma \geqq 0$. Hence

$$n = l\delta^3 + (P\gamma)^3 + \sum_{i=1}^{3} [(P^2+d_i)^3 + (P^2-d_i)^3],$$

where each cube is $\geqq 0$, since each $|d_i| < P^2$.

We may now readily verify that all integers of a wide range are sums of eight cubes. For $P > 1150$, (44) is satisfied if $C = 7.00001$. Take $n = Cm^6$. Then (43) gives

$$rm < P \leqq m, \quad r = (C/12)^{1/6}, \quad \log r = \bar{1}.9609862.$$

Start with $m=1500$. Then $rm=1371.1$. The ten primes $\equiv 2 \pmod 3$ between 1371 and 1500 are

$$1373, 1409, 1427, 1433, 1439, 1451, 1481, 1487, 1493, 1499.$$

Equating the fourth to rm', we get $m'=1567.7$. Hence the last seven primes serve for every m from 1500 to m'. Repeating with m' in place of m, we get as further P's 1511, 1523, 1553, 1559. Hence 1487, 1493, 1499, and these four serve for every m from m' to 1626.7. We advance similarly to 1705.5, 1751.4, and $M=1771.2$. But the four primes between M and the seventh prime 1733 serving for the third interval are all $\equiv 1 \pmod 3$. We may employ 41^2 and the last six of the seven primes, since their product by 41 exceeds the n corresponding to M, since (43) holds when $P=41^2$, and since Lemma 1 holds when p is replaced by any product P of primes each $\equiv 2 \pmod 3$. Hence we advance from M to $1637/r=1790.9$, and thence to 1823.7 (again using 41^2), 1856.5, $\mu=1869.6$. Lacking new primes $\equiv 2 \pmod 3$, we use $P=11 \cdot 167$ and note that the product of 11 and the last six of the seven primes exceeds the n corresponding to μ. We therefore advance to 1882.7. The next 13 steps proceed to 3307.1 by means of primes only, the number of available new primes being 2, 1, 5, 7, 7, 6, 4, 9, 8, 7, 11, 12, 10 respectively.

We may also proceed from 1500 to smaller values of m. Without new device, we reach 1163. For the next step we have available only five primes 1091, 1097, 1103, 1109, 1151, and $P=5 \cdot 227$, $11 \cdot 101$, $23 \cdot 47$. The advance to $1061/r=1160.7$ requires the verification that the integers n in the interval

which are divisible by the five primes and one of the factors of each of the three P's are actually sums of 8 cubes. With occasionally a like verification, we may advance in 26 steps to 821. The next step would involve serious additional verifications, since there are available only 761, 773, 797, 809, $11 \cdot 71$, $17 \cdot 47$ as values of P.

The n corresponding to the final $m=821$ is $10^{17}(21.436)$. Employing technical theory of primes, Baer* proved that every integer $>23 \cdot 10^{14}$ is a sum of eight cubes. The interest of our work lies in its very elementary character.

By two applications of Lemma 6 with $t=1$, $B=8043$, we find that every integer between 8043 and 227, 297, 300 is a sum of eight cubes. This limit is nearly 4% larger than that obtained by Lemma 3.

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* *Beiträge zum Waringschen Problem*, Dissertation, Göttingen, 1913.

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CUBIC CURVES AND DESMIC SURFACES; SECOND PAPER*

BY

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1. **Introduction.** It is evident superficially that there is some connection between cubic curves and desmic surfaces. It is well known that the equation of every cubic curve of the sixth class can be reduced to the form

$$y^2 = 4x^3 - g_2 x - g_3 = 4(x - e_1)(x - e_2)(x - e_3), \tag{1}$$

and the coördinates of a point on this curve are given parametrically as $x = \wp(u)$, $y = \wp'(u)$, where $\wp(u)$ is Weierstrass's elliptic $\wp$-function which is a solution of the differential equation

$$[\wp'(u)]^2 = 4[\wp(u)]^3 - g_2\wp(u) - g_3. \tag{2}$$

To every cubic of genus 1 there corresponds such a $\wp$-function and to every $\wp$-function there corresponds a projective class of cubics. On the other hand, the desmic surface may be defined analytically as the locus of a point whose coördinates are

$$x_0 = \frac{\sigma(u)}{\sigma(v)}, \qquad x_1 = \frac{\sigma_1(u)}{\sigma_1(v)}, \qquad x_2 = \frac{\sigma_2(u)}{\sigma_2(v)}, \qquad x_3 = \frac{\sigma_3(u)}{\sigma_3(v)}, \tag{3}$$

where

$$\left(\frac{\sigma_i(u)}{\sigma(u)}\right) = \wp(u) - e_i. \tag{4}$$

Thus to every desmic surface corresponds a $\wp$-function and to every such function a class of projective desmic surfaces.

This superficial analytic indication of relationship between the classes of curves and surfaces sets the problem of finding intimate geometrical connections. I have shown some of these relations in a former paper† and now present others.

2. **Setting of the problem.** We recall first some known facts. (i) From an arbitrary point A on a cubic curve C_6^3 of the sixth class four tangents can

* Presented to the Society, April 2, 1926; received by the editors in January, 1927.

† These Transactions, vol. 28 (1926) pp. 502–522.

be drawn. The cross ratios of this pencil of tangents are constant as A describes the curve, and they are the mutual ratios of the roots of $z^2-3z+2/I^{1/2}=0$ where I denotes the absolute invariant $64S^3/(64S^3+T^2)$ of the cubic, or equally well of the corresponding $\wp$-function. (ii) If ABC are three collinear points on C_6^3, then the three quadrangles of the points of contact of the tangents from them form a $(12_4, 16_3)$ Hessian configuration on the curve. Any two of the quadrangles are perspective from each vertex of the third. (iii) Three tetrahedra are in *desmic* formation when their vertices form a $(12_4, 16_3)$ space configuration, any two of the tetrahedra being perspective from each vertex of the third. (iv) They form the base of a pencil of quartic surfaces, called *desmic surfaces.* The points are the 12 nodes and the lines of perspectivity are the 16 lines on each surface. (v) The pencil contains three degenerate surfaces, namely the tetrahedra. If $\lambda=0$, $\mu=0$, $\nu=0$ be the equations of these three surfaces, then $\lambda+\mu+\nu\equiv 0$. (vi) If these forms be evaluated for an arbitrary point $P(x)$ of space, not on the tetrahedra, then the equation of the desmic surface D through P may be written

$$\lambda_x(y_0^2y_1^2+y_2^2y_3^2)+\mu_x(y_0^2y_2^2+y_3^2y_1^2)+\nu_x(y_0^2y_3^2+y_1^2y_2^2)=0. \tag{5}$$

(vii) The tetrahedra are also the base of a net of quadrics and the generators of these quadrics constitute a desmic cubic complex of lines. (viii) Now, a point $P(x)$ determines a desmic surface of the pencil and is the vertex of a cubic cone of the complex. As shown in the former paper, this cone passes through the vertices of the tetrahedra and so cuts an arbitrary transversal plane in a cubic curve with the desmic points projecting into a Hessian configuration on it. Moreover, the tangent plane to D at P cuts the cone in the three generators which give on C_6^3 the three collinear points ABC proper to the Hessian configuration. Conversely, it was shown in the first paper (p. 509) how to construct a set of tetrahedra and a desmic surface to correspond to a given C_6^3. When one of the desmic tetrahedra is taken for reference and a vertex of a second for unit point, then the equation of the cubic curve on $y_3=0$ is

$$\begin{aligned}(x_0^2-x_3^2)y_1y_2(x_2y_1-x_1y_2)+(x_1^2-x_3^2)y_2y_0(x_0y_2-x_2y_0)&\\ +(x_2^2-x_3^2)y_0y_1(x_1y_0-x_0y_1)=0.^*&\end{aligned} \tag{6}$$

3. **Developments for the general case.** We seek some of the consequences of these properties. We compute the absolute invariant of the cubic curve by Salmon's formulas, and find

* Loc. cit. equation 11.

$$I = \frac{4(\mu\nu + \nu\lambda + \lambda\mu)^3}{4(\mu\nu + \nu\lambda + \lambda\mu)^3 + (\mu - \nu)^2(\nu - \lambda)^2(\lambda - \mu)^2}. \tag{7}$$

Thus the invariant of the cubic curve and of the cubic cone is expressed in terms of the coefficients of the surface. The cross ratios of the four tangent planes through a generator of a cone, or of the four tangent lines which they cut on the transversal plane, are of the type form $-\lambda_x : \mu_x$; and as these ratios must be the same for all points of D (cf. equation (5)) it follows that

All cones of a desmic cubic complex whose vertices lie on one desmic surface have the same absolute invariant and are projective.

4. The assignment of $\sigma u/\sigma v$ as x_0, etc. is arbitrary. For a given $P(x_i)$, 23 other points may be obtained by permuting the coördinates, and the 24 correspond to the symmetric group G^4. This group contains as subgroup the symmetric group G_6^3. We find that the 24 points lie by fours on six distinct desmic surfaces of the pencil and the absolute invariant is the same for all six surfaces, for λ, μ, ν are merely permuted by the permutations of the G_6. Conversely, the absolute invariant, when equated to an arbitrary number, gives an equation of the 24th degree which factors into six of the fourth degree, and these give the six conjugate desmic surfaces of the pencil. Hence

The locus of the vertices of the projective cubic cones in a desmic cubic complex consists of six conjugate desmic surfaces.

5. The cross ratios $-\lambda:\mu$ have a still closer relation to the features of the surface. Those generators of the cone which lie in the tangent plane at P are three of the bitangents which can be drawn from P to D, and their second points of contact can be determined as follows. If the vertices of two of the tetrahedra be taken as the eight invariant points of a cubic involution $x' = 1/x$, then D transforms into itself and P interchanges with P', the second point of contact on one of the bitangents. If tetrahedra II and III be the invariant base of the transformation, we shall say that the bitangents signalized in this manner are *of the first system.* Now the four planes determined on $PP'A$ by the vertices of tetrahedron I have cross ratios of type $-\lambda:\mu$. Hence

The bitangents of the same system on a desmic surface subtend at the vertices of the proper tetrahedron four planes of constant cross ratio.

6. By a theorem of Steiner's these lines also cut the faces of the tetrahedron in a range of the same cross ratio. Thus

Bitangents of the same system on a desmic surface belong to a complex of Reye (tetrahedral complex) for which the corresponding tetrahedron of nodes is fundamental.

Under a cubic involution whose eight invariant points are the vertices of two desmic tetrahedra, a desmic surface transforms into itself and the lines which join corresponding points belong to a tetrahedral complex on the third tetrahedron.

7. **Special surfaces.** When the value of I is 1, 0, or ∞, the corresponding cubic curve is *harmonic*, *equianharmonic*, or *nodal*. We determine the corresponding desmic surfaces.

Let $I=1$, then $T^2=0$; thus the desmic surfaces on which the vertices of the cubic cones lie when the cubics are harmonic are

$$\mu_y - \nu_y = 0\,, \quad \nu_y - \lambda_y = 0\,, \quad \lambda_y - \mu_y = 0\,.$$

These are obtained by setting each factor of T equal to zero and considering x as a variable y. As $T^2=0$, each factor counts twice and the set of six desmic surfaces reduces to three.

8. If $I=0$, then $S^3=0$. The solution of the system

$$S = \rho(\mu\nu + \nu\lambda + \lambda\mu) = 0,$$

$$\lambda + \mu + \nu \equiv 0,$$

gives

$$\lambda = \lambda, \quad \mu = \omega\lambda, \quad \nu = \omega^2\lambda,$$

where ω is a complex cube root of unity. Hence the desmic surfaces for equianharmonic cubics are

$$y_0^2 y_1^2 + y_2^2 y_3^2 + \omega(y_1^2 y_3^2 + y_2^2 y_0^2) + \omega^2(y_2^2 y_1^2 + y_0^2 y_3^2) = 0$$

and

$$y_0^2 y_1^2 + y_2^2 y_3^2 + \omega^2(y_1^2 y_3^2 + y_2^2 y_0^2) + \omega(y_2^2 y_1^2 + y_0^2 y_3^2) = 0.$$

These are the only two distinct desmic surfaces in the set of six, for all the others obtained by permutation of the coefficients can also be obtained in this instance by multiplying each of the equations by ω and ω^2.

9. If $I=\infty$, then $S^3+64T^3=0$. This corresponds to $g_2^3-27g_3^2=0$ for the elliptic $\wp$-function. If the four numbers of the set (x^2) be taken as the roots of the quartic

$$f(z) \equiv a_0 z^4 + 4a_1 z^3 + 6a_2 z^2 + 4a_3 z + a_4 = 0,$$

then $g_2^3-27g_3^2$ is the discriminant of that equation and when equated to zero implies that at least two of the roots are equal. Therefore point (x) is on one of the six pairs of planes $y_i^2-y_j^2=0$. These are just the degenerate pairs of planes which taken again in twos make the three degenerate desmic

surfaces of the pencil. Thus the locus of (x) for a nodal cubic consists of the three degenerate desmic surfaces of the pencil:

$$\lambda = 0, \quad \mu' = 0, \quad \nu = 0.$$

Each of the corresponding nodal cubics degenerates to a line and a conic.

10. For cuspidal cubics T and S are zero simultaneously. We find that the corresponding cubics degenerate to three concurrent lines.

11. From these last two results we see that there is no correspondence between proper desmic surfaces and proper singular cubics; or otherwise, *no cone in a desmic complex has a single double line.*

12. **Associated surfaces of order eight.** It happens that the eighteen edges of the desmic tetrahedra are also the edges, in a different grouping, of a counter-set of desmic tetrahedra, and there is another pencil of desmic surfaces D' on these (First paper, p. 507). Through an arbitrary point P_x there passes a surface of each pencil. As any one surface D is cut by every surface of the other pencil, the values of the absolute invariant for the two surfaces through a point are different, in general. If we seek the locus of points for which they are equal, that is, form the combinations of type $\lambda_x\mu_x' - \lambda_x'\mu_x = 0$, we find six surfaces of the eighth order whose coefficients are *numerical.* A typical equation is

$$(x_0^2 - x_1^2)(x_2^2 - x_3^2)(\sum x_0^4 - 2\sum x_0^2 x_1^2 + 8x_0x_1x_2x_3)$$
$$+ 16x_0x_1x_2x_3(x_0^2 - x_2^2)(x_3^2 - x_1^2) = 0.$$

Thus we have a set of surfaces intimately connected with a set of desmic tetrahedra and its counter-set. Each surface passes through the 16 lines of the pencil $\{D\}$ and the 16 of pencil $\{D'\}$. Moreover, each passes through six of the eighteen edges of the tetrahedra.

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POSSIBLE ORDERS OF TWO GENERATORS OF THE ALTERNATING AND OF THE SYMMETRIC GROUP*

BY

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It is well known that every alternating and every symmetric group can be generated by two of its substitutions, and that two such generating substitutions can usually be selected in a large number of different ways. Since two operators of order 2 must always generate a dihedral group it is evident that no alternating group can be generated by two of its substitutions of order 2, and that the only symmetric group which can be thus generated is the symmetric group of order 6. On the other hand, it is known that with very few exceptions, relating to groups whose degrees do not exceed 8, every alternating group and every symmetric group can be generated by two of its substitutions of orders 2 and 3 respectively.† In the present article we shall prove that whenever an alternating group involves a substitution of order $l>3$ then it contains two substitutions of orders 2 and l respectively which generate the entire group. We shall also determine the degrees of all the symmetric groups to which a similar theorem does not apply.

Before proving this general theorem, it may be desirable to consider the more elementary question of generating an alternating or a symmetric group by two of its substitutions which are separately composed of a single cycle. When neither of the two numbers l_1, l_2 exceeds n but their sum exceeds n it is obvious that two substitutions s_1, s_2 which are separately composed of a single cycle, and whose orders are l_1, l_2 respectively, can be so selected that they generate a transitive group of degree n, and that half the substitutions of this transitive group are negative whenever at least one of the two numbers l_1, l_2 is even. If s_1 and s_2 do not have all their letters in common we may suppose that the common letters are arranged in the same order in both of these substitutions and hence their commutator is either of the form abc or of the form $ab \cdot cd$. If both of them are of degree n we may suppose that all their letters are arranged in the same order with the exception that two adjacent letters are interchanged. Hence their commutator is of the form abc in this case. The group generated by s_1, s_2 is obviously multiply transitive

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† G. A. Miller, Bulletin of the American Mathematical Society, vol. 7 (1901), p. 424.

whenever $n>3$ and hence it must be alternating or symmetric, at least when $n>8$, since the class of a primitive group which is neither alternating nor symmetric must exceed 4 whenever its degree exceeds 8. When n does not exceed 8 it is easy to verify directly that two such substitutions can be so selected as to generate the alternating groups when both of the numbers l_1, l_2 are odd, and the symmetric group when at least one of these numbers is even. These results may be stated in the form of a theorem as follows: *If l_1, l_2 represent a pair of numbers, each being greater than unity, such that neither exceeds n but their sum exceeds n, then it is always possible to find two cycles of orders l_1 and l_2 respectively such that they generate the alternating group of degree n whenever both l_1 and l_2 are odd. When at least one of these two numbers is even they generate the symmetric group of the same degree.*

From this theorem it results directly that if l_1, l_2 represents any pair of positive integers such that each exceeds unity then it is always possible to find two cycles s_1, s_2 of orders l_1, l_2 respectively such that the group generated by s_1, s_2 is either alternating or symmetric and has an arbitrary one of l_1, $l_1 \leqq l_2$, different degrees. For instance, pairs of cycles of order 9 can be selected such that each pair generates an arbitrary one of nine different alternating groups, viz., the alternating groups whose degrees vary from 9 to 17 inclusive, while an arbitrary one of the nine different symmetric groups of the degrees 10 to 18 can be generated by a cycle of order 9 and a cycle of order 10. This constitutes a complete solution of the elementary problem of generating alternating or symmetric groups by means of two cycles on their letters.

For the proof of the general theorem noted in the first paragraph it will often be convenient to use the following obvious theorem: *If a transitive group of degree n contains a cycle of prime order p, where p satisfies the condition $n/2<p\leqq n-3$, it must be either alternating or symmetric.* It is known that whenever $n>7$ it is always possible to find such a prime number.*

It will also frequently be desirable to use the following theorem:

If a transitive group is generated by two substitutions s_1, s_2 and if one of these substitutions s_2 involves one and only one cycle of a given prime degree while the other does not transform all the letters of this cycle into letters which do not occur in it, then the transitive group generated by s_1, s_2 is either alternating or symmetric whenever its degree exceeds the degree of this cycle by more than 2.

Since it is well known that a primitive group of degree n which does not include the alternating group of this degree can not involve a substitution

* G. A. Miller, School Science and Mathematics, vol. 21 (1921), p. 874.

composed of a single cycle of prime degree less than $n-2$, it is only necessary to show that s_1 and s_2 generate a primitive group. To do this we may transform by s_1 a power of s_2 which is equal to this prime cycle and thus obtain a prime cycle which has not all its letters in common with the former but has at least one letter in common therewith. Hence the group G generated by s_1, s_2 involves two such prime cycles which generate a doubly transitive group whose degree is just one larger than the degree of one of these cycles. Since a transitive group which contains a primitive subgroup of lower degree is itself primitive unless all the letters of this primitive subgroup appear in one of the sets of every one of its possible systems of imprimitivity, it results almost directly that G must be primitive and hence the theorem in question has been established.

To exhibit the nature of the limitations imposed in this theorem and at the same time prove a somewhat striking theorem it may be noted here that *if s_2 represents a substitution of composite order k_1k_2 and is composed of two cycles of orders k_1k_2 and k_1 respectively, and if s_1 is a transposition which involves one letter from each of these two cycles, then the group generated by s_1, s_2 is imprimitive and contains invariantly the direct product of k_1 symmetric groups which are separately of degree k_2+1.* A proof of this theorem results from the following consideration. The commutator of s_1 and $s_2^{k_1}$ is a cycle of order 3. The symmetric group of degree 3 generated by this commutator and s_1 has two letters in common with a cycle of $s_2^{k_1}$ and these two letters are adjacent in this cycle. Hence G involves the symmetric group of degree k_2+1 in view of the theorem noted near the close of the preceding paragraph. This symmetric group is transformed by s_2 into k_1 symmetric groups such that no two of them have a letter in common. The order of G is k_1 times that of the direct product of these symmetric groups. The simplest illustration of such a group is the transitive group of degree 6 and of order 72. In this case $k_1=k_2=2$.

Another elementary theorem which will be very useful in the solution of our general problem may be stated as follows:

If s_1, s_2 generate a transitive group and if s_1 has only one letter in common with some power of s_2 and if the letter by which this common letter is replaced in s_1 does not appear in a cycle of s_2 whose order is a multiple of the number of cycles in the given power of s_2, then this transitive group includes the alternating group of its degree.

The proof of this theorem is similar to that noted in the preceding paragraph. The commutator of the given power of s_2 and s_1 is again a cycle of

order 3 and hence G involves the alternating group on a number of letters which is at least one larger than the order of a cycle in the said power of s_2. This alternating group involves letters from at most two cycles of s_2. Hence s_2 would have to transform it into alternating groups on sets of distinct letters if the theorem were not true. As this is impossible from the conditions noted in the theorem it results that the theorem is established. When the given power of s_2 is a single cycle it is clear that the condition as regards the letter by which the common letter is replaced in it may be omitted in the theorem.

In what follows s_2 will represent a substitution of order $l>3$ while s_1 will represent a substitution of order 2. The smallest possible degree of s_2 is the sum of the highest powers of the prime power factors of l, and when l is even, s_2 must be negative both for this smallest degree and also for the next larger degree of the symmetric group in which it appears. In every larger symmetric group there is a positive as well as a negative substitution of order l. It will be assumed that s_1 and s_2 have been so constructed that they generate a transitive group of degree n, and it results directly from the theorems noted above that when l is a given even number, s_1, s_2 can be so chosen that they generate the symmetric group when n has the smallest possible value or the next larger value. In what follows we may therefore always assume that n has a larger value. When l is odd, s_2 must be positive, and when l is even it will be assumed that s_2 is positive unless the contrary is stated, and that the degree of s_2 is not less than $n-p_1+1$, where p_1 is the smallest prime factor of l when l is either odd or divisible by 3. When neither of these two conditions is satisfied then the degree of s_2 may be assumed to be at least $n-3$. Moreover, it will generally be assumed that the cycles of s_2 appear in descending order of magnitude in case there is a difference in their orders.

When l is divisible by at least three distinct prime numbers it is obvious that s_1 can be so selected that it has only one letter in common with the firse cycle of s_2 and that all the other cycles may be assumed to be of lower primt power orders. Moreover, when it is desirable to add to s_1 another transposition in order to give it the suitable sign we may form this, in case the order of the first cycle is divisible by the square of a prime number, on letters of this first cycle in such a way that the commutator of this cycle and s_1 is composed of a cycle of order 3 and of two transpositions. The square of this commutator is therefore a cycle of order 3 and may be used just as the commutator was used in the preceding case. When the order of this first cycle is not divisible by the square of a prime number, a transposition on the letters of the first cycle of s_2 may be added arbitrarily. Hence s_2 and s_1 can

always be so selected as to generate either the alternating group of degree n or the symmetric group of this degree, as may be desired, whenever l is divisible by at least three distinct prime numbers and the group concerned contains a substitution of order l. When l is divisible by two distinct odd prime numbers, or by one such number and 4, the remarks which have just been made still apply, and hence we may assume in what follows that l is either a power of a prime number or the double of a power of an odd prime number.

When $l=2p_1$, where $p_1>3$ is a prime number, it results again directly from the preceding theorems that s_1 and s_2 can be so selected as to generate either the symmetric or the alternating group, as may be desired, whenever the group in question involves a substitution of order l. When $l=2p_1^\alpha$, $\alpha>1$, and p_1 any odd prime number, it is easy to prove that s_1, s_2 can be so chosen that their product contains a cycle of order p, as defined above, and that another transposition can be added to s_1 so as to give it the proper sign without affecting this cycle of order p. A simple proof of this fact may be given as follows. First, select s_1 so as to connect the last letter of the first cycle of s_2 with the first letter of the second cycle, and the first letter of every other cycle with the second letter of the preceding cycle. When the degree s_2 is not n, we connect also the last letter of s_2 with a letter not found in s_2, and when n exceeds the degree of s_2 by more than 1 we may connect the additional letter or letters with the second or the second and third letter of the first cycle of s_2. It was noted above that there could not be more than two such additional letters. When s_1' is selected in this way it is obvious that $s_1's_2$ is a single cycle of order n. If the $(p+1)$th letter of this cycle, counting from the next to the last letter of the first cycle of s_2, is not in s_1', we add to s_1' the transposition composed of this letter and the next to the last letter of the first cycle in s_2. The product of s_2 and the s_1 thus obtained will then involve a cycle of order p and this cycle will not be affected by adding a properly chosen transposition to s_1 to give it the desired sign.

It remains to consider the case when the $(p+1)$th letter of the given cycle of order n appears in s_1'. If in this case the letter which precedes this $(p+1)$th letter does not appear in s_1' we start our cycle with the third letter from the end of the first cycle of s_2 and proceed as before. If both the $(p+1)$th letter and the preceding letter of the cycle of order n appear in s_1' and $p_1>3$ we start our cycle with the sixth letter from the end of the first cycle of s_2 and proceed as before. If $p_1=3$ we start with the fourth letter from the end of this cycle. If s_2 has been so chosen that it involves as small a number of transpositions as possible, no other case can present itself and hence it remains

to consider only the cases when $l=6$, and when l is a power of a single prime number.

When $l=6$ and s_2 is positive, $n>6$. When $n=7$ or 8, it follows directly from the general theorems noted above that s_1, s_2 can be suitably selected. When $n>7$ it may be assumed that the first cycle in s_2 is of order 6 and that s_2 involves no more than one transposition. It may also be assumed that its degree is not less than $n-2$, since cycles of order 3 may be added to it if necessary to increase its degree. Hence it is obvious that s_1 and s_2 may be so selected that their product involves a cycle of order p and that s_1 is either positive or negative as may be desired. It remains to consider the case when l is a power of a single prime number and $l>3$.

The case when $l=p_1$, p_1 being a prime number, is especially interesting since there is an infinite number of values of n, one and only one for each such prime number, such that the symmetric group of degree n contains a substitution of order l but cannot be generated by two operators of orders 2 and l respectively. The fact that $2p_1-1$ is such a value of n is obvious since s_1 must then be of degree $2p_1-2$ and hence it must be positive. This substitution and s_2 generate the alternating group of degree n according to the general theorems noted above, and hence it remains to prove that for every other value of n it is possible to find two substitutions of orders 2 and l respectively which generate either the alternating group or the symmetric group of degree n as may be desired. When $n<2p_1$, this requires no further proof since it is included in a general theorem noted above. When $n\geqq 3p_1$ it is obvious that s_1, s_2 can be so chosen that their product involves a cycle of order p. When $n=2p_1+k$, $k<p_1$, we may first consider the case when $k=p_1-1$. If $p>n-p_1'$, it must be at least equal to $n-p_1+2$. Hence we may connect the letters of the second cycle of s_2 by means of s_1' with the k letters which do not appear in s_2 and add a suitable transposition to s_1 so as to obtain a cycle of order p in the product of s_1' and s_2. An additional transposition on the letters of the first cycle of s_2 may be added to s_1' so as to give it the desired sign without affecting this cycle of order p, since p cannot exceed $n-3$. When $p=n-p_1$, then we may connect one of the letters not found in s_2 with the first letter in the first cycle of s_1 and the other letters not found in s_2 with letters of the second cycle of s_2. When $p<n-p_1$ we can evidently proceed in a similar way. Finally, when $k<p_1-1$ we may again connect by s_1 the letters of the second cycle of s_2 with those not found in s_2 and thus obtain suitable forms for s_1 and s_2. Hence the following theorem. *If $l>3$ represents a prime number which divides the order of the symmetric group of degree $n\neq 2l-1$, then it is always possible to find two operators of orders l and 2 respectively which generate this symmetric group and also two such operators*

which generate the alternating group of this degree. When $n=2l-1$ it is possible to find two such generators of the alternating group of degree n but it is impossible to find two such generators of the symmetric group of this degree.

When $l=p_1^\alpha$, where p_1 is any odd prime number and $\alpha > 1$, the substitution s_2 may be assumed to be of degree $n-k$, $k<p_1$. If s_2 involves more than one cycle we may again suppose that s_1' connects the last letter of the first cycle of s_2 with the first letter of the second cycle and the first letter of every other cycle with the second letter of the preceding cycle. Moreover, s_1' connects letters of the last cycle in s_2 with the k letters which do not appear in s_2. The product s_2s_1 is a cycle of degree n, and if the $(p+1)$th letter of this cycle, counting from the next to the last letter of the first cycle of s_2, does not appear in s_1' we adjoin to s_1' a transposition composed of this letter and the next to the last letter in the first cycle of s_2. If the said $(p+1)$th letter appears in s_1', the preceding letter cannot have this property, and hence we begin our cycle with the third letter from the end of the first cycle and adjoin to s_1' the transposition composed of this letter and the pth letter in the said cycle of order n. In both cases we obtain a value of s_1' such that s_2s_1' involves a cycle of order p, and that we can add another transposition to s_1', in case we desire to change its sign, without affecting this cycle of order p. When s_2 involves only one cycle, the k letters which do not appear in s_2 may be connected with the first k letters of s_2 and we may begin our cycle of order n with the last letter of s_2. The $(p+1)$th letter of this cycle cannot now appear in s_1' and hence we may adjoin to s_1' the transposition composed of this letter and the last letter of s_2 in order to obtain a cycle of order p in the product of s_2s_1; and an additional transposition can be added to this s_1 without affecting this cycle. When l is a power of 2 which is divisible by 8, similar considerations obviously apply, and hence it remains only to consider the case where $l=4$.

While an infinite number of exceptions presented themselves when $l>3$ was assumed to be an odd prime number, one for each such prime, there is only one exception when $l=4$, since every alternating group which involves substitutions of order 4 can be generated by two operations of orders 2 and 4 respectively, and every symmetric group, except the symmetric group of degree 6, can be generated by two such operators whenever its order is divisible by 4. To prove this theorem it may be assumed that s_2 is positive and that its degree is $n-k$, $k<2$, and that s_2 involves at most three transpositions. When $k=1$ the letter which is not found in s_2 is connected by s_1' with the last letter of s_2, while s_1' connects the other letters of s_2 as in the preceding cases so that s_2s_1' is again a single cycle of order n. When $n>19$, s_2 involves at least four cycles of order 4, and when $k=0$, s_1' is positive when the transposition is adjoined to give a cycle of order p in s_2s_1'. Hence when

$p=n-3$ it is not necessary to adjoin to s_1' an additional transposition to obtain generators of the alternating group. Generators of the symmetric group in this case may be obtained by replacing a cycle of order 4 in s_2 by two transpositions. When $p<n-3$ it is clearly possible to assume that s_2 is always positive and to adjoin a transposition to s_1' without affecting the cycle of order p. Hence two substitutions of orders 2 and 4 respectively can always be found so that they generate either the alternating or the symmetric group of degree n as may be desired whenever $n>19$.

When $7<n<20$ the value of p can be so selected as to make the determination of two possible generators s_1, s_2 very simple. The groups of degrees 6 and 7 are so well known that it seems unnecessary to give here two substitutions of orders 2 and 4 respectively which generate the alternating groups of these degrees or the symmetric group of degree 7 as may be desired. On the other hand, it may be of some interest to give an outline of a proof that the symmetric group of degree 6 cannot be thus generated. If it could be generated by two such substitutions we may assume that s_2 would be one of the following two substitutions, $abcd$, $abcd \cdot ef$. Since the separate groups generated by these substitutions are transformed into themselves by a group of order 16 on these letters we need to use for s_1 only one of each set of conjugates under this group. When s_2 is the former of the two given substitutions s_1 may therefore be assumed to be one of the following four substitutions:

$$ce \cdot df, \quad be \cdot df, \quad ab \cdot ce \cdot df, \quad ac \cdot be \cdot df.$$

In the first case the commutator of s_2 and s_1 would be $adcef$. This commutator and s_2^2 generate the simple group of order 60 since their product is of order 3. As this group is invariant under s_1 and s_2, these two substitutions generate the triply transitive group of order 120. The second and fourth substitutions to be used for s_1 evidently generate an imprimitive group with s_2, while the third and s_2 again generate the triply transitive group of order 120, since s_1s_2 in this case is $acedf$ and the square of this into s_2^2 is again of order 3. Hence these two substitutions generate the simple group of order 60 which is invariant under s_1 and s_2.

When $s_2=abcd \cdot ef$ it may be assumed that s_1 is one of the following three substitutions:

$$de, \quad ab \cdot cf \cdot de, \quad ac \cdot bf \cdot de.$$

In the first case it follows from a general theorem noted above the s_1 and s_2 generate a group of order 72. In the other two cases it is obvious that they must also generate an imprimitive group. Hence it has been proved that the symmetric group of degree 6 can not be generated by two of its substitutions of orders 2 and 4 respectively. That is,

Every symmetric group whose order is divisible by 4 *except the symmetric group of degree* 6 *can be generated by two operators of orders* 2 *and* 4 *respectively, and every alternating group which involves operators of order* 4 *can be generated by two such operators.*

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OPTICS IN HYPERBOLIC SPACE*

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1. Introduction

With Riemann we define the metric of H-space† by

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2}{\left[1 - \frac{x_1^2 + x_2^2 + x_3^2}{4R^2}\right]^2} = 16R^2 \frac{d\sigma^2}{\lambda^2}, \tag{1}$$

where

$$r^2 = x_1^2 + x_2^2 + x_3^2, \quad d\sigma^2 = dx_1^2 + dx_2^2 + dx_3^2, \quad \lambda = 4R^2 - r^2. \tag{2}$$

H-straights are defined by

$$\delta \int ds = 0. \tag{3}$$

In order to have a model of this space in which we can see the figures employed we may regard the x_1, x_2, x_3 as rectangular cartesian coördinates. Then the images of points of H_3 are points within the e-sphere $\lambda = 0$ which we call the λ-sphere. In this model H-straights are e-circles cutting $\lambda = 0$ orthogonally.

It is convenient to introduce new variables

$$z_i = \frac{4R^2}{\lambda} x_i, \quad i = 1, 2, 3; \quad z_4 = R(\mu/\lambda), \tag{4}$$

where

$$\mu = x_1^2 + x_2^2 + x_3^2 + 4R^2.$$

H-planes are defined by a linear relation

$$a_1 z_1 + a_2 z_2 + a_3 z_3 + a_4 z_4 = 0.$$

In the model they are e-spheres cutting $\lambda = 0$ orthogonally. The intersection of two H-planes are H-straights. H-planes through the origin O are also e-planes and the same is true of straights. The H-angle between two curves or surfaces or a curve and a surface is the same as the corresponding angle

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† For H- read hyperbolic; for e- read euclidean.

in the model. Figures may be moved about freely in H-space as in e-space. The coördinates z satisfy

$$(5)\qquad \{z^2\} = z_1^2 + z_2^2 + z_3^2 - z_4^2 = -R^2.$$

Besides the points so far considered we have certain ideal points, viz., those lying on the λ-sphere; they are at an infinite distance away from any ordinary point. Their z coördinates satisfy the relation

$$(6)\qquad \{z^2\} = 0.$$

The four planes $z_1=0, z_2=0, z_3=0, z_4=0$ form a tetrahedron, the plane $z_4=0$ being imaginary. It may be represented diagramatically by Fig. 1. The vertex A_k is opposite $z_k=0$. All straights perpendicular to $z_k=0$ meet in the vertex A_k. The displacement

$$z_1' = z_1, \quad z_2' = z_2 \cosh + z_3 \sinh \theta,$$
$$z_3' = z_2 \sinh \theta + z_3 \cosh \theta, \quad z_4' = z_4$$

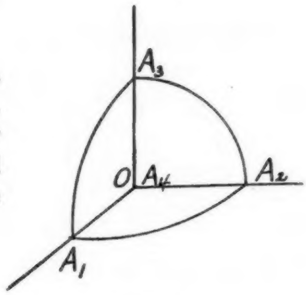

FIG. 1

defines a rotation θ about A_1. It leaves the plane $z_1=0$ unaltered, a figure in this plane being merely moved into a congruent figure.

2. Reflection and refraction on a plane surface

We shall suppose that the path of a ray of light in a heterogeneous medium satisfies

$$(1)\qquad \delta \int n\,ds = 0,$$

where n, the index, is a function of the coördinates x. When n = constant this becomes

$$(2)\qquad \delta \int ds = 0,$$

i.e., the path of a ray of light is a straight.

Consider two media of indices n, n' separated by an H-plane. A ray issuing from A arrives at A'. It is easy to see that the path lies in a plane normal to the boundary. We suppose it lies in the x, y plane. We must choose B so that $nAB+n'BA'$ or $s=np+n'p'$ is a minimum. Then

$$(3)\qquad \frac{ds}{dx} = n\frac{dp}{dx} + n'\frac{dp'}{dx} = 0.$$

Let $CB=x$, $BC'=x'$, $x+x'=c$, a constant, and therefore $dx+dx'=0$. Then

$$\cosh(p/R) = \cosh(a/R)\cosh(x/R), \quad \cosh(p'/R) = \cosh(a'/R)\cosh(x'/R);$$

therefore

$$\frac{dp}{dx} = \cosh(a/R)\frac{\sinh(x/R)}{\sinh(p/R)}, \qquad \frac{dp'}{dx} = -\cosh(a'/R)\frac{\sinh(x'/R)}{\sinh(p'/R)}.$$

These in (3) give

$$(4) \qquad n\cosh(a/R)\frac{\sinh(x/R)}{\sinh(p/R)} = n'\cosh(a'/R)\frac{\sinh(x'/R)}{\sinh(p/R)}.$$

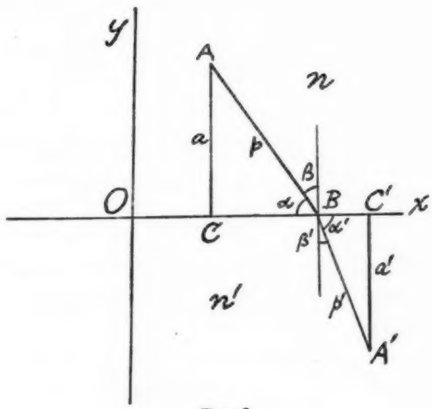

FIG. 2

But

$$\cos\alpha = \frac{\tanh(x/R)}{\sinh(p/R)}, \qquad \cos\alpha' = \frac{\tanh(x'/R)}{\sinh(p'/R)}.$$

These in (4) give

$$n\cosh(a/R)\frac{\cos\alpha}{\cosh(a/R)} = n'\cosh(a'/R)\frac{\cos\alpha'}{\cosh(a'/R)},$$

therefore

$$(5) \qquad n\cos\alpha = n'\cos\alpha' \quad \text{or} \quad \frac{\sin\beta}{\sin\beta'} = \frac{n'}{n},$$

which is the law of sines as in e-geometry.

In case of reflection, $n = n'$; we find in similar manner that the angle of incidence equals the angle of reflection.

We see at once the truth of the following

THEOREM. *The image of an object obtained by reflection on a plane mirror is the congruent figure back of the mirror and at the same distance as the object in front of it.*

Let us consider now refraction, on a plane surface, say $z_1 = 0$.

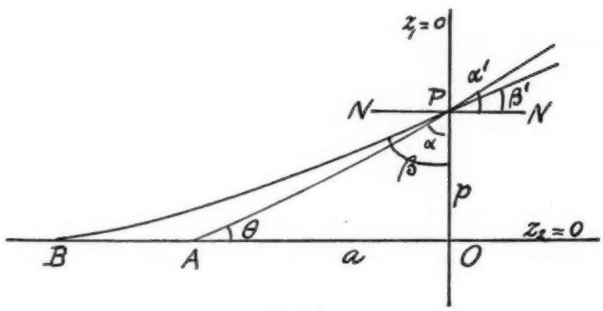

FIG. 3

In Fig. 3 a ray issues from A, and strikes the boundary surface at P, making the angle α' with the normal PN. The refracted ray PB' makes the angle β' with the normal, where

$$\frac{\sin \beta'}{\sin \alpha'} = n < 1, \text{ say.} \tag{6}$$

Produced backwards, it cuts $z_2 = 0$ at B. We set

$$\alpha + \alpha' = 90^\circ, \quad \beta + \beta' = 90^\circ, \quad AO = a, \quad BO = b, \quad PO = p$$

in H-measure. Set also

$$C = \cosh(a/R), \quad S = \sinh(a/R), \quad T = \tanh(a/R).$$

Then

$$\tan\theta = \frac{\tanh(p/R)}{S}, \qquad \tan\alpha = \frac{T}{\sinh(p/R)}\cdot$$

Thus

$$\tan\alpha' = \frac{C\tan\theta}{(1 - S^2\tan^2\theta)^{1/2}}, \tag{7}$$

while β' is given by (6).

Let us suppose θ is small; then (7) gives

$$\tan \alpha' = C\theta/(1 - S^2\theta^2)^{1/2} = C\theta, \text{ or } \alpha' = C\theta,$$

neglecting θ^2. Thus

$$\beta' = n\alpha' = nC\theta.$$

Next we have

$$\sinh (p/R) = \frac{S \tan \theta}{(1 - S^2 \tan^2 \theta)^{1/2}} = S\theta,$$

(8)

$$\tanh (b/R) = \frac{\sinh (p/R)}{\tan \beta'} = \frac{S \cdot \theta}{nC\theta} = \frac{1}{n} \tanh (a/R),$$

a constant, neglecting higher powers of θ.

Suppose now we revolve Fig. 3 about OA. The symmetry of the figure gives the following

THEOREM. *A nearly normal pencil of rays issuing from A forms a virtual image at B, at a distance b given by* (8).

Since $\tanh (b/R) \leqq 1$ we see that when a is such that

(9) $$\tanh (a/R) \geqq n, \quad n < 1,$$

the ray PB does not meet the axis OA. Hence

THEOREM. *When the pencil enters a denser medium no image is formed if A is at a distance a satisfying* (9).

A small rotation θ about the vertex A_1 opposite the plane $z_1 = 0$ in Fig. 4 moves O to O' and LM to $L'M'$, such that $A'O' = AO$, $BO = B'O'$. As $A'O'$ is normal to the plane $z = 0$, we see the rays from A' behave in the same manner as those from A. Thus they meet in B'; hence

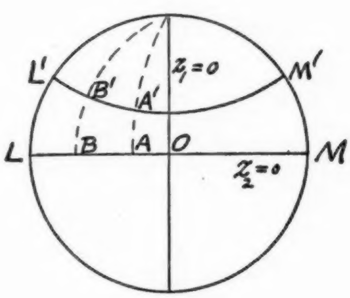

FIG. 4

THEOREM. *The small figure AA' has BB' as its image.*

Let δ be the distance of A' to LM and δ' the distance of B' to this line. Then

$$\sinh (\delta/R) = \cosh (\rho/R) \sin \theta, \quad \rho = OA'.$$

Similarly

$$\sinh(\delta'/R) = \cosh(\rho'/R)\sin\theta, \quad \rho' = OB'.$$

Hence

$$m = \frac{\sinh(\delta'/R)}{\sinh(\delta/R)} = \frac{\cosh(\rho'/R)}{\cosh(\rho/R)}.$$

This we may call the magnification of the image.

By (8)

$$\tanh(\rho'/R) = (1/n)\tanh(\rho/R).$$

Thus

$$m = \frac{n}{[n^2\cosh^2(\rho/R) - \sinh^2(\rho/R)]^{1/2}} = \frac{n}{[1-(1-n^2)\cosh^2(\rho/R)]^{1/2}}.$$

This becomes imaginary if

$$\frac{1}{1-n^2}\cosh^2(\rho/R) > 1, \quad \text{or} \quad \tanh(\rho/R) > n.$$

It is not difficult to prove the

THEOREM. *If the pencil is not narrow, the rays issuing from A do not meet in a point.*

3. CENTRAL OPTICAL IMAGERY

As a first approximation to the path of light through a system of lenses the theory of collineation was shown by Maxwell and Abbe to be of extreme value. We shall now show that H-optics does not have this elegant tool to

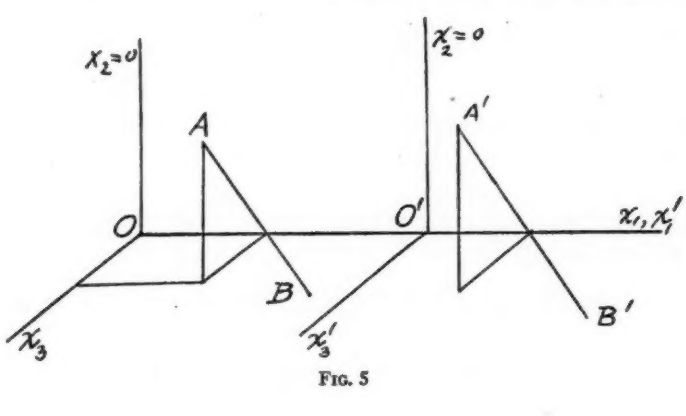

FIG. 5

work with. We suppose that we are dealing with a symmetrical optical system whose axis is the x_1-axis. The collineation defined by this system must have the form

$$
(1) \qquad \begin{aligned}
z_1' &= a_{11}z_1 + a_{12}z_2 + a_{13}z_3 + a_{14}z_4, \\
& \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
z_4' &= a_{41}z_1 + a_{42}z_2 + a_{43}z_3 + a_{44}z_4.
\end{aligned}
$$

Let A, B be two symmetric points in the object space relative to the axis x_1, and A', B' their images. If the coördinates of A are (z_1, z_2, z_3, z_4), the coördinates of B are $(z_1, -z_2, -z_3, z_4)$ while the coördinates of B' are $(z_1', -z_2', -z_3', z_4')$. These in (1) give

$$
(2) \qquad \begin{aligned}
z_1' &= a_{11}z_1 - a_{12}z_2 - a_{13}z_3 + a_{14}z_4, \\
-z_2' &= a_{21}z_1 - a_{22}z_2 - a_{23}z_3 + a_{24}z_4, \\
-z_3' &= a_{31}z_1 - a_{32}z_2 - a_{33}z_3 + a_{34}z_4, \\
z_4' &= a_{41}z_1 - a_{42}z_2 - a_{43}z_3 + a_{44}z_4.
\end{aligned}
$$

Comparing (1), (2) we get

$$
a_{12} = a_{13} = 0, \quad a_{21} = a_{24} = 0, \quad a_{31} = a_{34} = 0, \quad a_{42} = a_{43} = 0.
$$

Thus (1) reduces to

$$
(3) \qquad z_1' = a_{11}z_1 + a_{14}z_4, \qquad z_2' = a_{22}z_2 + a_{23}z_3, \qquad z_3' = a_{32}z_2 + a_{33}z_3, \\
z_4' = a_{41}z_1 + a_{44}z_4.
$$

Since the collineation is central we may restrict ourselves to a plane, say the x_1x_2 plane. We may thus write (3)

$$
(4) \qquad z_1' = az_1 + bz_3, \quad z_2' = cz_2, \quad z_3' = \alpha z_1 + \beta z_3.
$$

Since the coördinates z' must satisfy the relation §1, (5), we find that

$$
(5) \qquad a^2 - \alpha^2 = 1, \quad b^2 - \beta^2 = -1, \quad c^2 = 1, \quad ab = \alpha\beta.
$$

Hence

$$
(6) \qquad z_1' = az_1 + bz_3, \quad z_2' = \delta z_2, \quad z_3' = \epsilon bz_1 + \epsilon az_3,
$$

where $\delta^2 = \epsilon^2 = 1$.

Now the equations (6) define a displacement , i.e. the image is merely a congruent figure of the object. Hence

THEOREM. *If the image afforded by a central optical system is the result of a collineation, the image is an exact replica of the object without magnification.*

If the coördinates of an ideal point are set in (6) we find $\{z'^2\}=0$; hence

THEOREM. *As a point A recedes to infinity, its image A' does the same.*

4. REFLECTION ON A SPHERE

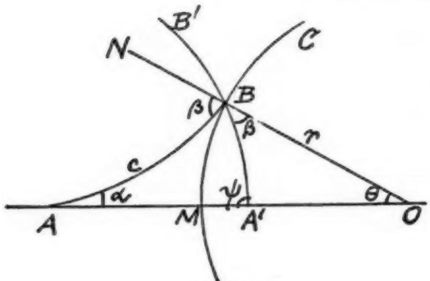

FIG. 6

If a ray issuing from A meets the spherical mirror C at a point B, it is reflected along BB':

$$OA = a, \quad OB = r, \quad AB = c, \quad OA' = a',$$

in H-measure. In the triangle OAB,

$$\cosh(c/R) = \cosh(a/R)\cosh(r/R) - \sinh(a/R)\sinh(r/R)\cos\theta, \tag{1}$$

$$\sin\beta = \frac{\sinh(a/R)}{\sinh(c/R)}\sin\theta. \tag{2}$$

In the triangle OBA',

$$\cos\psi = \cos\beta\cos\theta - \sin\beta\sin\theta\cosh(r/R), \tag{3}$$

$$\sinh(a'/R) = \frac{\sin\beta}{\sin\psi}\sinh(r/R). \tag{4}$$

These equations give the path of the reflected ray.

The H-length of the arc $MB=l=R\theta\sinh(r/R)$. We set $l\leqq l_0$, $\theta\leqq\theta_0$, and suppose θ_0 and l_0/R are small. We will assume that r/R is small while a/R is large. Then approximately

$$\cosh(a/R) = \sinh(a/R) = \tfrac{1}{2}e^{a/R}.$$

Then (1) gives

$$\cosh(c/R) = \tfrac{1}{2}e^{a/R} = \cosh(a/R)\text{ ; therefore } a = c.$$

This in (2) gives

$$\beta = \theta.$$

Thus (3) gives

$$\cos\psi = \cos^2\theta - \sin^2\theta\cosh(r/R)$$
$$= 1 - \theta^2 - \theta^2\left(1 + \frac{r^2}{2R^2}\right) = 1 - 2\theta^2 = 1 - \frac{(2\theta)^2}{2}\cdot$$

Hence

$$\sin\psi = 2\theta, \quad \text{or} \quad \psi = 2\theta.$$

This in (4) gives

$$\sinh (a'/R) = \frac{\theta}{2\theta} \sinh (r/R) = \tfrac{1}{2}(r/R),$$

therefore

$$a'/R = \tfrac{1}{2}(r/R) \quad \text{or} \quad a = \tfrac{1}{2}r.$$

Hence rays issuing from A meet at a fixed point A'.

Let us now rotate Fig. 6 through a small angle about O. The point A describes the arc AP while its virtual image describes the arc $A'P'$. Hence

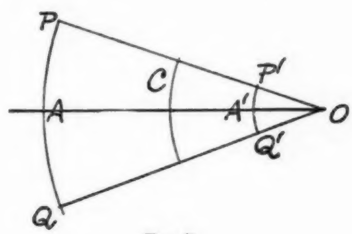

FIG. 7

THEOREM. *The image $P'Q'$ of a small distant object PQ in a small mirror of radius r, small compared with the space constant R, is virtual and lies at the distance $\frac{1}{2}r$ from the center of the mirror.*

5. REFRACTION ON SPHERICAL SURFACES

In Fig. 8, a ray issues from O meeting the spherical surface at B and makes the angle α with the normal BN. It is bent into BH which produced

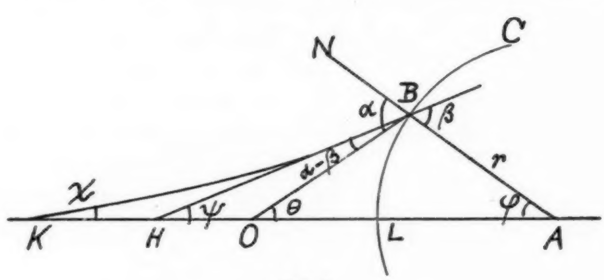

FIG. 8

backwards meets OA at K. Its tangent at B is BH:

$$OA = a, \quad OB = s, \quad OH = h, \quad \text{rad}\, AB = r, \quad \text{in } e\text{–measure};$$

$$OK = \eta, \quad OB = \sigma, \quad OA = \omega, \text{ in } H\text{–measure}.$$

Then

$$\text{(1)} \qquad \tan\theta = \frac{r\sin\phi}{a - r\cos\phi}, \qquad \theta = \beta + \psi = \alpha - \beta + \psi,$$

$$\text{(2)} \qquad \sin\beta = n\sin\alpha,\ n < 1,\ \alpha = \theta + \phi,$$

$$\text{(3)} \qquad s = r\frac{\sin\phi}{\sin\theta}, \qquad h = \frac{s\sin(\alpha-\beta)}{\sin\psi}.$$

In the H-triangle OBK,

$$\text{(4)} \qquad \cos\chi = \cos(\alpha-\beta)\cos\theta + C\sin(\alpha-\beta)\cos\theta,$$

$$\text{(5)} \qquad \sinh(\eta/R) = \frac{\sin(\alpha-\beta)}{\sin\chi}\cdot S,$$

where

$$\text{(6)} \qquad C = \cosh(\sigma/R), \qquad S = \sinh(\sigma/R).$$

These equations give the refracted ray.

We suppose now that ϕ is small. From (1) we have, neglecting small quantities of higher order,

$$\text{(7)} \qquad \theta = \frac{r\phi}{a-r}.$$

From (2),

$$\text{(8)} \qquad \alpha = \theta + \phi = \frac{a\phi}{a-r},$$

$$\text{(9)} \qquad \alpha/\theta = a/r, \quad \text{a constant}.$$

From (2),

$$\text{(10)} \qquad \beta = n\alpha = \frac{na\phi}{a-r}, \qquad \alpha - \beta = (1-n)\alpha = m\alpha, \qquad m = 1 - n.$$

From (3),

$$\text{(11)} \qquad s = r\phi/\theta = a - r.$$

Thus s and hence σ, its H-measure, is constant. From (4),

$$\cos\chi = 1 - \tfrac{1}{2}(m^2\alpha^2 + \theta^2 - 2m\alpha\theta C) = 1 - \tfrac{1}{2}\theta^2X^2,$$

where

$$X^2 = m^2(\alpha^2/\theta^2) + 1 - 2mC(\alpha/\theta);$$

therefore, using (9),

$$X^2 = m^2(a^2/r^2) + 1 - 2mC(a/r), \quad \text{a constant.} \tag{12}$$

Thus

$$\sin \chi = \theta \cdot X.$$

From (9), (10) and (15)

$$\sinh (\eta/R) = \frac{(\alpha - \beta)S}{\theta X} = \frac{ma}{r} \frac{S}{X}, \quad \text{a constant.} \tag{13}$$

Hence

THEOREM. *Neglecting small quantities of order* >1, *rays issuing from O and meeting a convex spherical surface nearly normally, meet at the virtual image K at a distance* η *given by* (13)

For χ to be real, X must $\geqq 0$, or

$$C = \cosh (\sigma/R) \leqq \frac{r^2 + m^2a^2}{2amr}. \tag{14}$$

Now as $a \to 2R$, $\sigma \to \infty$ while the right side is finite, we have the

THEOREM. *When the source moves away beyond a certain distance given by* (14) *there is no image.*

We note that when the = sign holds in (14) the point K is at ∞.

Let us displace Fig. 8 by a rotation about the pole of OA, so that A coincides with the origin O. Let the source be now at A, Fig. 9, and its image at C.

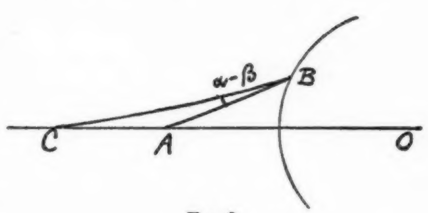

FIG. 9

Since distances and angles have remained unaltered we have as before $AB = \sigma$, $OK = \eta$, and relation (5) still holds.

Let us now revolve Fig. 9 about O through a small angle τ. Then A and C describe small arcs AA', CC' of lengths

$$\delta = R\tau \sinh (\omega/R), \quad \delta' = R\tau \sinh [(\omega + \eta)/R].$$

We may regard as the magnification of the object the quotient

$$\frac{\delta'}{\delta} = \frac{\sinh [(\omega + \eta)/R]}{\sinh (\omega/R)}. \tag{15}$$

Finally let us note that the point H is fixed. For from (3)

$$h = \frac{(\alpha - \beta)s}{\psi} = \frac{ms\alpha}{\psi} = \frac{ms\alpha}{\theta - m\alpha},$$

which is constant since α/θ is, by (9). We note that when the source of light recedes to infinity the point H is real.

6. Lenses

In practical optics when one wishes to calculate the best shapes of a new system of lenses designed to achieve a certain object, it is necessary in the end to make laborious trigonometric calculations. In H-space it seems necessary to do this even when one wishes only approximate results since the theory of collineation does not apply. As in the preceding sections, so here, we make a favorable choice of the origin, sometimes using e-measure and sometimes H-measure, angles having the same measure in both geometries.

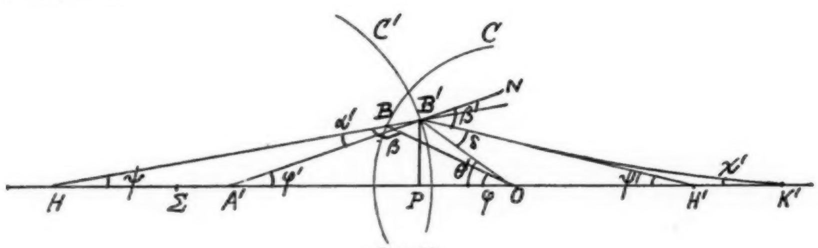

Fig. 10

In Fig. 10 we suppose the center of the lense C is at the origin O while the center of C' is at A'. The source of light is at Σ. A ray meets C at B and is bent into BB', which as before we may suppose to be an e-straight. At B it is bent into $B'K'$ whose e-tangent at B' is $B'H'$. The incident ray at B' makes the angle α', the emergent ray makes the angle β' with the normal $B'N'$ at B'. Then

$$\sin \beta' = \frac{1}{n} \sin \alpha'. \tag{1}$$

We set

$$\beta = OBH, \quad \psi = B'HA', \quad \phi' = B'A'O, \quad \phi = BOA', \quad \theta' = B'OA',$$
$$\psi' = B'H'O, \quad \chi' = B'K'O;$$

$$OH = h,\ OH' = h',\ OA' = a',\ OB' = s',\ l = A'H = h - a',$$

$$OB = r,\ A'B' = r',\ \text{in } e\text{-measure};$$

$$\sigma' = OB',\quad \eta' = OK' \text{ in } H\text{-measure};$$

$$C' = \cosh(\sigma'/R),\qquad S' = \sinh(\sigma'/R).$$

We saw in §5 that H is fixed when ϕ is small. We have now

$$\sin\alpha' = (l/r')\sin\psi, \tag{2}$$

$$\tan\theta' = \frac{r'\sin\phi'}{a' - r'\cos\phi'}, \tag{3}$$

$$s' = \frac{r'\sin\phi'}{\sin\theta'}. \tag{4}$$

In the H-triangle $OB'K'$, $\delta = OB'K' = \theta' + \phi' - \beta'$, and

$$\cos\chi' = \cos\delta\cos\theta' + C'\sin\delta\sin\theta', \tag{5}$$

$$\sinh(\eta'/R) = S'\frac{\sin\delta}{\sin\chi'}. \tag{6}$$

Suppose again that ϕ is small. Then

$$\psi = \beta - \phi = n\alpha - \phi,\quad \alpha' = l\psi/r', \tag{7}$$

$$\phi' = \alpha' + \psi = (l + r')\psi/r', \tag{8}$$

$$\theta = \frac{r'\phi'}{a' - r'} = \frac{l + r'}{a' - r'}\psi = g\psi,\quad g \text{ constant}, \tag{9}$$

$$s' = r'\phi'/\theta' = (a' - r')/r',\quad \text{a constant}. \tag{10}$$

Thus neglecting small quantities of order >1, s' and hence σ', also C', S' are constants. From (5),

$$\begin{aligned}\cos\chi' &= (1 - \delta^2/2)(1 - \theta'^2/2) + 2C'\delta\theta' \\ &= 1 - \tfrac{1}{2}\theta'^2(1 + \delta^2/\theta'^2 - 2C'\delta/\theta') = 1 - \tfrac{1}{2}\theta'^2X^2,\end{aligned}$$

therefore

$$\sin\chi' = \theta'X,\quad \chi' = \theta'X.$$

From (6)

$$\sinh(\eta'/R) = \frac{\delta}{\theta'}\cdot\frac{S'}{X}. \tag{11}$$

Now

$$\delta/\theta' = 1 + \phi'/\theta' - \beta'/\theta',\quad \beta' = \frac{\alpha'}{n} = \frac{l\psi}{nr'};$$

therefore

$$\beta'/\theta' = \frac{l}{nr'} \cdot \frac{\psi}{\theta'} = \frac{l}{gnr'}, \quad \text{a constant,}$$

whence δ/θ' is constant, and therefore by (11), η' is constant.

In order that χ' be real, we must have $X^2 \geqq 0$ or

$$(\delta/\theta' - C')^2 \geqq C'^2 - 1 = S'^2 \text{ or } \delta/\theta' \geqq C' \pm S' = \Delta = e^{\pm\sigma'/R},$$

or

$$a' - \frac{l}{gn} \geqq \Delta \text{ or } na' - \frac{a' - r'}{1 - r'/l} \geqq n\Delta. \tag{12}$$

When r'/l can be neglected, this gives

$$r' \geqq (1 - n)a' + n\Delta. \tag{13}$$

Hence the

THEOREM. *If the conditions* (12) *or* (13) *are satisfied, rays issuing from a point and meeting a convex lens nearly normally will unite in a conjugate point K' determined by* (11).

7. BOUGUER'S THEOREM

We suppose that the index of refraction n at a point $x_1x_2x_3$ is a function of the distance of the point from the origin O. The path of a ray in this medium is determined by

$$\delta \int n\, ds = 0.$$

We have

$$\delta(n\, ds) = n \cdot \delta\, ds + ds \cdot \delta n, \qquad \delta n = \sum_\alpha \frac{\partial n}{\partial x_\alpha} \delta x_\alpha,$$

$$\delta \cdot ds = \frac{16R^2}{\lambda^2} \sum_\alpha \frac{dx_\alpha}{ds} \delta \cdot dx_\alpha + \frac{2}{\lambda^2} \sum_\alpha x_\alpha \delta x_\alpha,$$

$$\int_a^b \frac{n}{\lambda^2} \frac{dx_\alpha}{ds} \delta \cdot dx_\alpha = -\int_a^b \delta x_\alpha \frac{d}{ds}\left(\frac{n}{\lambda^2} \frac{dx_\alpha}{ds}\right) ds,$$

therefore

$$\delta \int_a^b n\, ds = \int_a^b ds \sum_\alpha \left\{ \frac{\partial n}{\partial x_\alpha} + \frac{2nx_\alpha}{\lambda^2} - 16R^2 \frac{d}{ds}\left(\frac{n}{\lambda^2} \frac{dx_\alpha}{ds}\right) \right\} \delta x_\alpha.$$

Thus the equations of the path are

$$\frac{\partial n}{\partial x_\alpha} + \frac{2nx_\alpha}{\lambda^2} - 16R^2\frac{d}{ds}\left(\frac{n}{\lambda^2}\frac{dx_\alpha}{ds}\right) = 0 \qquad (\alpha = 1,2,3). \tag{1}$$

The matrix

$$\begin{pmatrix} x_1, & x_2, & x_3 \\ \dfrac{dx_1}{ds}, & \dfrac{dx_2}{ds}, & \dfrac{dx_3}{ds} \end{pmatrix} \tag{2}$$

has three determinants D_α. From (1) we find at once that

$$\frac{d}{ds}\left(\frac{n}{\lambda^2}\cdot D_\alpha\right) = 0.$$

Hence

$$\frac{n}{\lambda^2}\cdot D_\alpha = a_\alpha, \quad \text{a constant}.$$

If we multiply these three equations by x_1, x_2, x_3 we get $a_1x_1+a_2x_2+a_3x_3=0$. Hence

THEOREM. *The path of a ray in this medium lies in an H-plane through O.*

In our model, the radius vector r from O to the point $P(x_1, x_2, x_3)$ has as direction cosines $l_\alpha = x_\alpha/r$, while those of the ray are $m_\alpha = dx_\alpha/d\sigma$. If θ is the angle between r and the ray,

$$\cos\theta = \sum l_\alpha m_\alpha.$$

Now

$$\frac{dx_\alpha}{ds} = \frac{\lambda}{4R}\frac{dx_\alpha}{d\sigma} = \frac{\lambda m_\alpha}{4R},$$

therefore

$$\cos\theta = \frac{4R}{r\lambda}\sum x_\alpha\frac{dx_\alpha}{ds}.$$

Here the sum on the right is the scalar product of the matrix (2). By Lagrange's theorem,

$$\sum D_\alpha^2 = \sum x_\alpha^2 \cdot \sum\frac{dx_\alpha^2}{ds^2} - \left(\sum x_\alpha\frac{dx_\alpha}{ds}\right)^2$$

or

$$\frac{\lambda^4}{n^2}\sum a_\alpha^2 = r^2\frac{\lambda^2}{16R^2}(1-\cos^2\theta) = r^2\frac{\lambda^2}{16R^2}\sin^2\theta.$$

Hence

$$\frac{nr}{\lambda}\sin\theta = \text{constant along the ray.} \tag{3}$$

This gives Bouguer's theorem in H_3-space.

THEOREM. *The path of a ray of light in a medium whose index is a function only of the distance from O satisfies* (3).

If ρ is the length of the vector OP in H-measure we have

$$r = 2R\tanh(\rho/2R), \quad \lambda = 4R^2\operatorname{sech}^2(\rho/2R), \quad \frac{r}{\lambda} = \frac{\sinh(\rho/R)}{4R}.$$

Thus (3) becomes

$$n\sinh(\rho/R)\sin\theta = c, \quad \text{a constant}, \tag{4}$$

where the quantities ρ, θ, η are now expressed in H-measure. From this relation we get the equation of the path of the ray. For

$$d\rho = \cos\theta\, ds, \qquad d\phi = \frac{(\sin\theta)\, ds}{R\sinh(\rho/R)},$$

therefore

$$\frac{d\rho}{d\phi} = \frac{\cos\theta}{\sin\theta}\cdot R\sinh(\rho/R),$$

or using (4)

$$\frac{d\phi}{d\rho} = \frac{1}{(n^2\sinh^2(\rho/R) - c^2)^{1/2}}. \tag{5}$$

Here n is a function of ρ only.

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GEODESICS ON SURFACES OF GENUS ZERO WITH KNOBS*

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INTRODUCTION

Poincaré has studied the geodesics upon closed surfaces of genus zero which are everywhere convex†. General surfaces of genus greater than one have been studied by H. M. Morse.‡ The surfaces considered in this paper are of genus zero and closed, but have upon them regions of negative curvature as well as regions of positive curvature. They are of such a nature that the removal of certain portions which we call knobs leaves a region with extremal-convex boundaries. A remarkable subset of the geodesics issuing from any point not on a knob is considered, and results are obtained resembling those of Hadamard for geodesics on surfaces of negative curvature.§

PART I. THE SURFACE

1. The surface defined. Let us consider a closed surface homeomorphic with a sphere. We assume that the surface can be divided into a finite number of overlapping regions, such that the cartesian coördinates x, y, z of the points of any of these regions can be expressed in terms of parameters u and v by means of functions with continuous derivatives of at least the fourth order and such that

$$\left[\frac{D(x,y)}{D(u,v)}\right]^2+\left[\frac{D(x,z)}{D(u,v)}\right]^2+\left[\frac{D(y,z)}{D(u,v)}\right]^2\neq 0.$$

It is assumed, moreover, that the surface possesses n knobs ($n>1$), where a ***knob*** will be defined as a finite portion K of the surface which is: (a) bounded by a closed curve C with continuously turning tangent; (b) homeomorphic with the interior and boundary points of a circle; (c) such that the geodesics

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* Presented to the Society, January 2, 1926; received by the editors in August, 1926.

† Poincaré, *Sur les lignes géodésiques des surfaces convexes*, these Transactions, vol. 6 (1905), p. 237.

‡ Morse, *A fundamental class of geodesics on any closed surface of genus greater than one*, these Transactions, vol. 26 (1924), pp. 49–60.

§ Hadamard, *Les surfaces à courbures opposées et leurs lignes géodésiques*, Journal de Mathématiques pures et appliquées, (5), vol. 4 (1896), p. 27.

tangent to the boundary C lie interior to K in the immediate neighborhood of the point of contact. Finally, the curves C bounding different knobs on the surface are assumed to have no points in common with each other.

Let now the knobs be cut from the body of the surface along the curves C. There will remain a surface S with n closed bounding curves. S will be *extremal-convex* in the sense* that, on the original uncut surface, geodesics tangent to the boundary of S lie outside of S in the immediate neighborhood of the point of contact. In this paper we consider geodesics in so far as they lie on S.

2. **Sufficient conditions for the existence of knobs.** If any point P on a regular surface is taken as origin of a system of geodesic polar coördinates, the element of arc takes the form

$$ds^2 = dr^2 + C^2(r,\phi)d\phi^2,$$

in which r is the distance measured from P along any geodesic through P, and ϕ is the angle between the tangent at P to this geodesic and that to an arbitrary geodesic through P. Also $C(0,\phi)=0$ and $C_r(0,\phi)=1$.†

Within the region where the geodesics $\phi=$constant form a field, all the geodesics not $\phi=$constant are solutions of the Euler equation for

$$s = \int [\dot{r}^2 + C^2(r,\phi)]^{1/2} d\phi,$$

namely

$$\ddot{r} - \left(\frac{2C_r}{C}\right)\dot{r}^2 - \left(\frac{C_\phi}{C}\right)\dot{r} - CC_r = 0.$$

For a geodesic which is tangent to the geodesic circle $r=r_0$, $\dot{r}=0$ at the point of contact. At this point, therefore,

$$\ddot{r} = CC_r. \tag{1}$$

In order that the simply connected region bounded by $r=r_0$ be a knob it is necessary that for each geodesic tangent to $r=r_0$, $\ddot{r}\leqq 0$ at the point of tangency. A sufficient condition is that $\ddot{r}<0$ and hence from (1), it is also sufficient that $C(r_0,\phi)>0$ and $C_r(r_0,\phi)<0$ for all values of ϕ.

We shall deduce a sufficient condition for the existence of a knob in terms of the total curvature.

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* Cf. G. D. Birkhoff, *Dynamical systems with two degrees of freedom*, these Transactions, vol. 18 (1917), p. 216.

† Cf. Darboux, *Surfaces*, III, p. 157.

Let any point P on a regular surface be taken as the origin of a system of geodesic polar coördinates (r, ϕ) and let $K(r, \phi)$ be the total curvature of the surface at the point (r, ϕ).

THEOREM 1. *If there exists a constant $r_0>0$ such that*

$$\left(\frac{\pi}{2r_0}\right)^2 < K(r,\phi) < \left(\frac{\pi}{r_0}\right)^2 \tag{2}$$

for any ϕ and for $r \leqq r_0$, then the set of points for which $r \leqq r_0$ form a knob.

By Gauss's theorem*

$$C_{rr}(r,\phi) + K(r,\phi)C(r,\phi) = 0.$$

Consider any value ϕ_0 of ϕ. We assume that (2) holds for $r \leqq r_0$. We shall compare the solutions of the differential equations

$$\frac{d^2x}{dr^2} + \left(\frac{\pi}{2r_0}\right)^2 x = 0, \tag{a}$$

$$\frac{\partial^2 C}{\partial r^2} + K(r,\phi)C = 0, \tag{b}$$

$$\frac{d^2z}{dr^2} + \left(\frac{\pi}{r_0}\right)^2 z = 0, \tag{c}$$

which have the initial conditions

$$x(0) = 0, \quad \frac{dx(0)}{dr} = 1\,;$$

$$C(0,\phi_0) = 0, \quad \frac{\partial C(0,\phi_0)}{\partial r} = 1\,;$$

$$z(0) = 0, \quad \frac{dz(0)}{dr} = 1.$$

By the use of Sturm's comparison theorem,† it follows from (b) and (c) and the inequality $K<(\pi/r_0)^2$, that $C(r, \phi_0)>z(r)>0$ within the interval $0<r\leqq r_0$. Similarly, from (a) and (b) and the inequality $(\pi/2r_0)^2<K$, we have $C(r, \phi_0)<x(r)$ for the same interval. Since $C(r, \phi_0)$ is bounded in the interval, it takes on a maximum for a value r_m of r where

$$0 < r_m \leqq r_0.$$

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* Cf. W. Blaschke, *Vorlesungen über Differential Geometrie*, I, p. 61.

† Bieberbach, *Theorie der Differentialgleichungen*, 1923, pp. 144–5.

We shall prove that

$$\frac{r_0}{2} < r_m < r_0.$$

Suppose first that $0 < r_m \leqq r_0/2$. From (b) and (c), omitting the arguments in the functions concerned, we find readily

$$C\ddot{z} - zC_{rr} = \left[K - \left(\frac{\pi}{r_0}\right)^2\right]Cz.$$

Integrating with respect to r from $r=0$ to $r=r_m$, we have

$$[C\dot{z} - zC_r]_0^{r_m} = \int_0^{r_m}\left[K - \left(\frac{\pi}{r_0}\right)^2\right]Cz\,dr.$$

Since $C(0, \phi_0) = z(0) = C_r(r_m) = 0$,

$$C(r_m)\dot{z}(r_m) = \int_0^{r_m}\left[K - \left(\frac{\pi}{r_0}\right)^2\right]Cz\,dr < 0.$$

But $C(r_m) > 0$ and $\dot{z}(r_m) \geqq 0$ for $r_m \leqq r_0/2$. Hence, a contradiction is obtained. Similarly, the assumption $r_m = r_0$ leads to a contradiction.

Therefore

$$\frac{r_0}{2} < r_m < r_0,$$

as stated. Hence $C(r_0, \phi_0) > 0$ and $C_r(r_0, \phi_0) < 0$ and by (1), $\bar{r} < 0$. Since (2) holds for any value ϕ_0 of ϕ, the theorem follows.

3. **Class A geodesic segments.** Consider now the surface S and denote the n extremal-convex boundaries by $c_1, c_2, \cdots, c_n$. If P and Q are any two points within or on the boundary of S, there exists on S joining P to Q at least one rectifiable curve whose length furnishes a minimum with respect to the lengths of all rectifiable curves on S connecting P and Q.* Such a minimizing arc is a geodesic segment with continuously turning tangent and has no points in common with any boundary c_k, with the possible exception of P and Q themselves. Such geodesic segments will be called *Class A geodesic segments on S.*

No two class A segments on S not joining the same points can intersect more than once. For suppose two such segments PQ and RS intersect in D and E. Then it follows from the definition of geodesics of class A that the segments DE of the two geodesics have the same length. In a portion of PQ

* Bolza, *Vorlesungen über Variationsrechnung*, 1909, pp. 422, 436.

including DE as an interior segment, the arc DE can be replaced by its equal segment on RS. The resulting curve, however, has corners and hence can be shortened,* contrary to the assumption that PQ was a class A segment.

4. **The covering surface and linear sets.** The surface S can now be rendered simply connected as follows: From any arbitrarily chosen point P on c_n, we can and will cut the surface along a system of class A geodesics $h_1, h_2, \cdots, h_{n-1}$, leading to points on $c_1, c_2, \cdots, c_{n-1}$, respectively. According to the result of the preceding paragraph, the geodesics h_k have no other points than P in common. We denote by T the simply-connected piece of surface obtained by cutting S along the h's.

We now consider M, the *universal covering surface*† of S, made up of an infinite number of copies of T. On M any two points or curves which overlie the same point or curve on S are said to be *congruent*. The boundaries of M are congruent to the boundaries $c_1, c_2, \cdots, c_n$ of S.

Let r be an integer, positive, negative or zero. Let T_r denote a particular copy of T on M. A *linear set*‡ of copies of T on M will be defined to be a region of M consisting of a set of the copies of T on M of the form

$$\cdots, T_{-2}, T_{-1}, T_0, T_1, T_2, \cdots, \tag{1}$$

or of the form of any subset of consecutive symbols of (1), in which each copy T_k of T is joined to the succeeding one along a common boundary and all copies are distinct. A linear set which has no first or last copy of T will be termed an unending linear set.

Part II. The class A geodesic rays through a point

5. **Unending geodesics of class A.** We have defined (§3) a geodesic segment joining a pair of points P and Q on S to be of *class A on S* if its length furnishes a minimum with respect to the lengths of all rectifiable curves *on* S joining P and Q. We shall now define similarly a geodesic segment joining a pair of points P and Q on M to be of *class A on M* provided its length furnishes a minimum with respect to the lengths of all rectifiable

* In the regular problem of the calculus of variations, the Erdmann corner point condition is never satisfied.

† Cf. H. M. Morse, *A one-to-one representation of geodesics on a surface of negative curvature*, American Journal of Mathematics, vol. 43 (1921), pp. 35–40; H. Weyl, *Die Idee der Riemannschen Fläche*, pp. 47–53; Kerékjártó, *Vorlesungen über Topologie*, I, pp. 158, 173–184.

‡ For another point of view, cf. Oswald Veblen, *Analysis Situs*, The Cambridge Colloquium, Part 2, Chapter V. The linear set used in our paper has, however, the important metrical property of being bounded by geodesic segments congruent to the boundaries of T, in addition to the abstract properties of the members of the Poincaré group.

curves *on* M joining P and Q. Every geodesic segment of class A on S is also of class A on M but the converse is not true.

Now any two points P and Q on M can be included in a region R consisting of a finite number of copies of T. The boundaries of R will be found in part among the boundaries of M, composed of segments congruent to the c_k's, and in part among the geodesic segments of class A on M congruent to the h_k's. The segments of the boundary of R congruent to the c_k's intersect the segments congruent to the h_k's so that the angles interior to R do not exceed π. The region R will still be extremal-convex in the sense of Birkhoff,* and it follows as in §3 that the points P and Q on R can be joined by a geodesic segment of class A on M which lies in R and has no points in common with the boundary of R with possible exception of P and Q themselves.

We shall now define an *unending geodesic of class A on M* as an unending geodesic lying entirely on M, every finite segment of which is a geodesic segment of class A on M.

It follows from a discussion similar to that of §3 that no two unending geodesics of class A on M can intersect more than once.

Now let g be an unending geodesic of class A on M. Clearly g cannot become infinite in length in any single copy of T. In leaving a copy of T, g cannot be tangent to any of the geodesic segments separating that copy of T from the remainder of M. Further, g can have only one point of intersection with any such geodesic segment. It follows that an unending class A geodesic on M is contained in one and only one unending linear set.

Henceforth, geodesics and geodesic segments will be assumed to be on M unless the contrary is stated.

By using an argument due to Morse† we obtain the following theorem:

THEOREM 2. *Given any unending linear set, there exists at least one unending geodesic of class A contained wholly in the given linear set.*

6. **Semi-infinite sets and geodesic rays.** Let T_0 be any copy of T on M. A semi-infinite linear set starting from T_0 will be defined by a sequence of copies of T that go to make up M, namely: $T_0, T_1, T_2, \cdots$, in which each copy T_k of T is joined to the succeeding one along a common boundary and all copies are distinct.

A geodesic ray of class A issuing from a point P will be defined to be a portion of a geodesic, in one sense unending and in the other stopping at P, and such that every finite segment is of class A. Corresponding to Theorem 2 we have the following theorem for a semi-infinite linear set.

* Loc. cit., p. 216.

† American Journal of Mathematics, loc. cit., pp. 47–48.

THEOREM 3. *Given a semi-infinite linear set starting from a copy T_0, there exists issuing from any point P of T_0 at least one geodesic ray of class A contained wholly in the given linear set.*

If the given surface has two knobs, there are only two semi-infinite linear sets beginning with any copy T_0 of T. If, however, the number of knobs exceeds two, it is readily shown that the number of such sets starting from a given T_0 has the power of the continuum.

By Theorem 3 there exists issuing from a point P on a given copy T_0 of T at least one geodesic ray of class A belonging to each semi-infinite linear set beginning with that copy. For $n>2$, therefore, there are through P a set of such geodesic rays in power equal to the power of the continuum.

Let the positive sense of any geodesic ray of class A through P be the sense that leads from P. The direction of any such geodesic ray will now be specified by the angle $\theta(-\pi<\theta\leqq\pi)$ measured in an arbitrary sense about P, between the positive tangent to it at P and the tangent at P to an arbitrary geodesic through P. We shall show that the infinite set of directions θ has a very remarkable subset which is perfect and nowhere dense.

7. **Special and general linear sets.** Any semi-infinite linear set L is topographically equivalent to one of the two regions of the plane bounded by two parallel straight lines and a transversal to them. Consider those boundaries B and B' of L which in this correspondence are topographically equivalent to the semi-infinite segments of the parallel lines.

Now all the semi-infinite linear sets whose first copy of T contains P will be divided into two classes:

(A) General linear sets;

(B) Special linear sets.

A semi-infinite linear set will be called *special* if either B or B', after at most a finite segment, consists entirely of a boundary of M made up of a succession of segments which are congruent to a single one of the boundaries c_k on S. Any set not special will be called *general*.

Consider any two semi-infinite linear sets L and L', beginning with T_0, and let them be represented by the sequences of copies of T: $T_0, T_1, T_2, \cdots$, and $T_0, T_1', T_2', \cdots$, respectively. If the successive copies $T_1', T_2', T_3', \cdots, T_n'$ are respectively the same as $T_1, T_2, \cdots, T_n$, but if T_{n+1}' is different from T_{n+1}, the two linear sets will be said to *diverge* after sharing n copies of T. Now if B and B' are the semi-infinite boundaries of L, it is clear that T_{n+1}' is joined to $T_n'=T_n$ along a segment q of either B or B' which arose from one of the pieces h_k. The set L' will be said to diverge from L along B or B' according as q lies on B or B'.

The general linear sets (A) and the special linear sets (B) may now be characterized with respect to diverging linear sets as follows: In the case of any general linear set L, there exist, corresponding to any positive integer m, linear sets which diverge from L along B and along B' after sharing with L more than m copies of T. In the case of a special linear set L, for sufficiently large values of m, all linear sets different from L and sharing with L more than m copies of T, diverge from L along either B or B', but not both.

8. **Boundary geodesic rays.** Consider now a general linear set L. Let $q_1, q_2, \cdots$ be successive segments of B which arose from cuts h_k. Let L_1 be a linear set which diverges from L along q_1, L_2 a linear set which diverges from L along q_2 and so on. On each linear set L_i, there exists at least one geodesic ray of class A issuing from P. Let $\theta_1, \theta_2, \cdots$ be the directions at P of geodesic rays of class A issuing from P, chosen on $L_1, L_2, \cdots$, respectively. The set of θ's has at least one limit angle Θ. Let g be the geodesic ray through P with the direction Θ. Then g belongs to a linear set, which must be L, and is itself of class A.

If θ is measured in a proper sense about P, it will be true that for all integers exceeding a suitably chosen integer m, $\theta_i < \theta_{i+1}$. Hence Θ will be the limit of an increasing sequence of the θ_i's.

It may well happen that there exist in some or all of the linear sets L_i more than one geodesic ray of class A issuing from P and belonging to L_i. It is conceivable that, if a different set of these geodesic rays were picked out, and correspondingly different angles θ_i^*, a different angle Θ^* would be approached as a limit. If it is remembered, however, that no two geodesic rays issuing from P can intersect, it will be seen that for a proper choice of $\theta = 0$, $\theta_{i-1} < \theta_i^* < \theta_{i+1}$. Hence $\Theta = \Theta^*$ and the geodesic ray determined by the limiting process is unique.

Similarly, let $q_1', q_2', q_3', \cdots$ be successive segments of B' which arose from cuts h_k. Let L_1' be a linear set diverging from L along q_1' and so on. If $\theta_1', \theta_2', \cdots$ are the directions at P of geodesic rays of class A lying on $L_1', L_2', \cdots$, the set of directions θ_i' has a limit Θ' which defines a geodesic ray g' through P. The geodesic ray g' is a ray of class A on L and does not depend upon the particular selection of geodesic rays, issuing from P, made from $L_1', L_2', \cdots$.

The geodesic rays g and g' bound a region R in L between which there can lie no geodesic rays of class A issuing from P except those remaining forever in L. Further if g'' be any geodesic ray of class A issuing from P and lying in L, and not g or g', it must lie in this region R. For otherwise g'' would cut all of the geodesic rays of L_i or else all of the geodesic rays of L_i' which issue from P with angles nearer Θ or Θ' respectively than the initial angle of g''.

The geodesic rays g and g' will be called the ***boundary geodesic rays*** of the set L. In case there is only one geodesic ray of class A on L, $g = g'$.

For special linear sets, we adopt a similar procedure except that only one boundary B or B' contains an infinite number of segments which arose from cuts h_k. Hence only one boundary geodesic ray is defined for a special linear set.

We shall consider in the following only the boundary rays of the semi-infinite linear sets whose first copy of T contains P.

9. Generalization of a theorem of Hadamard. We prove the following theorem.

THEOREM 4. *The set of the directions at P of all the boundary rays of class A issuing from any given point P of T_0 is perfect and nowhere dense.*

It follows from the process by which boundary rays are defined that the direction of each such ray is the limit of the directions of others of the same kind, and further that any limit direction of an infinite subset of directions of boundary rays is itself the direction of a boundary ray. Hence the given set is perfect.

Between the boundary rays of any given linear set (if there are two), there are, of course, geodesic rays with intermediate directions. Such rays are clearly not themselves boundary rays, since there are at most two boundary rays in any given linear set.

Consider, on the other hand, two boundary rays g and g' which belong to different linear sets L and L'. There exists a last copy of T, say T_n, which the two sets have in common. On the boundary of T_n, there exists between the pieces to which T_{n+1} of L and T'_{n+1} of L' are attached, in either cyclic order, at least one point Q on the boundary of M. Therefore a class A geodesic segment may be drawn from P to Q whose direction will be between the directions of the boundary rays g and g', and which will pass off M at Q. All rays through P with angles sufficiently near that of the geodesic segment PQ will also pass off M and hence cannot be boundary rays through P. Hence the set of directions of boundary rays through P is nowhere dense and the theorem is proved.

As a special case, this theorem reduces to a result obtained by Hadamard in his study of surfaces of negative curvature.* Let us suppose that the closed surface of genus zero with which we start is such that the removal of the knobs leaves a region S which is of negative curvature throughout. By virtue of this special condition, all geodesic rays issuing from a given point P on M

* Loc. cit., p. 69.

and remaining on M are of class A. It can be shown that there exists issuing from P, one and only one geodesic ray belonging to each semi-infinite linear set whose first copy of T contains P.

The theorem for the special case under consideration may now be expressed in a form not involving the covering surface, and becomes then the theorem of Hadamard:

If S is of negative curvature throughout and if P is any point on S, the set of the directions at P of the geodesic rays which issue from P and remain on S is perfect and nowhere dense.

Part III. Periodic geodesics

10. **Definitions.** We shall return to the consideration of unending linear sets. Let such a set be represented by the succession of copies of T,

$$(1) \qquad \cdots, T_{-2}, T_{-1}, T_0, T_1, T_2, \cdots.$$

The symbols of this sequence may be put into one-to-one correspondence with a succession of symbols

$$(2) \qquad \cdots, p_{-2}, p_{-1}, p_0, p_1, p_2, \cdots,$$

as follows. Let the $2(n-1)$ boundary pieces of T, which arise from the cuts $h_i(i=1, 2, \cdots, (n-1))$, be numbered in cyclic order, beginning with an arbitrary piece. These numbers will then be associated with the boundary pieces of each of the copies T_k occurring in (1). In the linear set (1), let p_k be the number of the boundary piece of T_k to which T_{k+1} is joined. Then (2) is the sequence obtained by replacing T_k by p_k.

Suppose now that (2) is found to consist, in both senses, of an unending repetition of a finite set G of successive symbols. The linear set will then be called *periodic* and G a *generator* of the set. Any generator of (2) which is made up of the smallest possible number of symbols will be called a *fundamental generator* of (2). Any succession of copies of T in (1) which corresponds to a generator of (2) will be called a *cycle* of the linear set. A *fundamental cycle* will then be a cycle which corresponds to a fundamental generator.

Those geodesic segments which separate successive cycles of the periodic linear set from each other will clearly be congruent.

In the representation (1) of a given unending linear set L, a copy T_k of T will be said to *occur n copies later* than another copy T_k' of T, if $k-k'=n$. We shall now define a transformation t of a periodic set L into itself, as follows: Suppose that each fundamental cycle of L is composed of n copies of T. Then under t every point of each copy T_k of T is to be replaced by its congruent point in that copy of T which in (1) occurs n copies later. Con-

gruent points under t will be spoken of as *congruent points one fundamental cycle apart* and congruent points under t^m as congruent points m fundamental cycles apart.

11. **Lemma on class A geodesic segments joining congruent points.** We prove the following lemma.

LEMMA 1. *A class A geodesic, joining congruent points m fundamental cycles apart on a periodic linear set, intersects itself on the original surface in such a manner, that if cut at these points of intersection, the resulting geodesic segments can be regrouped, reordered, and rejoined on M so as to make up m curve segments each joining congruent points one fundamental cycle apart.*

Let R be the region formed by $m>1$ successive fundamental cycles of a periodic linear set. Let C be a class A geodesic segment connecting congruent points A and B on the boundaries b_1 and b_2 of R which separate it from the remainder of the linear set. We shall first prove that there exist on C at least two congruent points one fundamental cycle apart. Let C' be the curve segment determined by all the points congruent to C under t, the transformation of the preceding paragraph. If C and C' have a point in common on R, the point in common on C, if considered also as a point on C', appears clearly congruent to a point on C one fundamental cycle previous. We must show then that C and C' do have a point in common on R. Now C divides R into two regions R_1 and R_2, having no other points in common than those which belong to C. We denote by $P_0, P_1, \cdots, P_m$ the intersections of C with the geodesic segments separating different cycles of R from each other and from the remainder of the linear set, the geodesic segments being taken in the order of progression along the set. C' does not intersect the first such boundary and extends beyond R in the positive direction of the set. Its intersections with the geodesic boundaries of successive cycles will be denoted by $P_1', P_2', \cdots, P_{m+1}'$.

If C and C' have a point in common, the proof is complete. In the contrary case, we assume that P_1' lies in R_2 but not on C. Then C' crosses the boundary of R_2 either along C or b_2. In the former case, the proof is again complete. In the later case, the point P_m' on b_2 lies in R_2. Let $s_1, s_2, \cdots, s_m$ be the distances along the geodesic segments separating cycles of the linear set, measured from the points where these segments enter R_2 from outside the linear set to $P_1, P_2, \cdots, P_m$, respectively. Let $s_1', s_2', \cdots, s_m'$ be similarly defined relative to $P_1', P_2', \cdots, P_m'$. We have then $s_m'<s_m$, by hypothesis. Since P_m' is congruent to P_{m-1}, we have $s_m'=s_{m-1}$ and hence $s_{m-1}<s_m$. Similarly, $s_{m-1}'<s_{m-1}$, whence $s_{m-2}<s_{m-1}$. Proceeding in this manner, we obtain the conditions

$$s_0 < s_1 < s_2 \cdots < s_{m-1} < s_m,$$

as the only possible case under which the proof might fail. But P_0 and P_m are congruent and therefore $s_0 = s_m$. Hence this set of conditions cannot be satisfied and C and C' have a point in common. There exist therefore on C two congruent points one fundamental cycle apart.

Let now P and Q be the points of C one fundamental cycle apart whose existence has just been proved. Let AP and QB be the segments of C which respectively precede and follow PQ in the order following the linear set. Now consider on R the curve C_1 which consists of the segment QB and a segment congruent to AP under the transformation t. Then C_1 joins points on R which are $(m-1)$ fundamental cycles apart and intersects each geodesic segment separating successive cycles once and only once. The procedure of the preceding paragraphs will now show that, if $m>2$, C_1 contains a subsegment joining congruent points one cycle apart. Repetition of the process m times gives the conclusion.

12. **Existence of periodic geodesics of class A.** *An unending periodic geodesic* g is defined as an unending geodesic composed of successive congruent segments. Such a geodesic necessarily lies on an unending periodic linear set L, and overhangs a closed geodesic on the surface S. A segment of g whose length is that of this closed geodesic on S will be called a *fundamental segment* of g.

LEMMA 2. *The length of a class A geodesic segment on M is a continuous function of its end points.*

Suppose PQ and $P'Q'$ are two class A geodesic segments such that the distance $PP' = e_1$ and the distance $QQ' = e_2$, where e_1 and e_2 are arbitrarily small positive numbers and the distances PP' and QQ' are measured along class A geodesic segments. If d and d' are respectively the lengths of PQ and $P'Q'$, it follows from the class A character of PQ that

$$d \leqq d' + (e_1 + e_2),$$

i.e.,

$$d' \geqq d - (e_1 + e_2).$$

Likewise from the class A character of $P'Q'$ it follows that

$$d' \leqq d + (e_1 + e_2).$$

Combining,

$$d - (e_1 + e_2) \leqq d' \leqq d + (e_1 + e_2).$$

The inequalities express the stated continuity property.

THEOREM 5. *Corresponding to any unending periodic linear set there exists on the set at least one unending periodic geodesic of class A.*

Let D be a fundamental cycle of the given linear set. Each pair of congruent points on the boundaries b_1 and b_2 separating D from the remainder of the linear set can be joined by a class A geodesic segment lying entirely in D. It follows from Lemma 2 that the length of such class A segments is a single-valued continuous function of an end point on either b_1 or b_2. Since the domain of such an end point is closed, there exists a segment AB whose length l_m gives a minimum among the lengths of all class A geodesics joining congruent points on b_1 and b_2. The segment AB will be proved to be a fundamental segment of a periodic geodesic of class A.

Choose arbitrarily a positive direction along b_1 and a corresponding congruent positive direction along b_2. Then AB makes the same angle with b_1 as it does with b_2. Otherwise, on the surface obtained by "healing" b_1 and b_2 together, AB would have a corner and could be shortened on this surface. Recutting the surface to form D, the shortened curve would connect congruent points on b_1 and b_2. The class A geodesic segment joining these same two congruent points would be shorter than AB, contrary to the supposition that AB was the shortest class A segment connecting congruent points on b_1 and b_2.

It now follows that in the given unending periodic set of which D is a fundamental cycle, the curves congruent to AB in the successive cycles join on to each other continuously to form an unending periodic geodesic g of which AB is a fundamental segment.

We shall prove by the aid of Lemma 1 that g is of class A.

Assume, on the contrary, that there exist on g two points P and Q which can be connected on M by a curve shorter than the segment PQ of g. Then there can be found on g two points A and B, m fundamental cycles apart, containing P and Q between them and lying on geodesic segments separating cycles of the linear set from each other, where m is a properly chosen positive integer.

On g, the segment AB has the length ml_m. Now join A and B by a geodesic segment C of class A and length C. By hypothesis, $C < ml_m$. Hence at least one of the m segments of C of Lemma 1 is necessarily less than l_m in length. It would follow that a pair of congruent points could be found on the geodesic boundaries of a fundamental cycle, for which the shortest connecting path on M would be in length less than l_m, contrary to the definition of l_m. Therefore g is of class A and the theorem is completely proved.

It follows from the theorem just proved that there exist on an extremal-

convex surface S with at least three boundaries, an enumerably infinite number of closed (or periodic) geodesics. In particular, there exist closed geodesics deformable into each of the boundaries of S.

13. **Class A geodesics of the same type.** In the remainder of the paper we will state without proof the consequences of the supposition that there exist two or more class A periodic geodesics of the same type, that is, belonging to the same linear set.

It is necessary that all such periodic geodesics of the same type have fundamental segments of the same length.

No two class A periodic geodesics, g_1 and g_2, of the same type can intersect, for if they intersected once, they would intersect an infinite number of times, contrary to a fundamental property of class A geodesics. Hence g_1 and g_2 separate a ribbon-like region R from the remainder of the set. We suppose that on R there exist no other periodic geodesics of class A.

THEOREM 6. *If a class A geodesic g lies completely on a region R bounded by two class A periodic geodesics g_1 and g_2 and there exist on R no class A periodic geodesics other than g_1 and g_2, then g is either asymptotic to g_1 in its positive sense and to g_2 in its negative sense or it is asymptotic to g_2 in its positive sense and g_1 in its negative sense, where the positive sense of g is taken to be that sense which follows the linear set to which g belongs.*

The results for geodesic rays are similar.

THEOREM 7. *There exist issuing from any point P between g_1 and g_2 at least four geodesic rays of class A, respectively*
(a) *positively asymptotic to g_1,*
(b) *negatively asymptotic to g_1,*
(c) *positively asymptotic to g_2,*
(d) *negatively asymptotic to g_2.*
Moreover, any geodesic ray of class A issuing from P belongs to one of these four classes.

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CONCERNING END POINTS OF CONTINUOUS CURVES AND OTHER CONTINUA*

BY

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1. Introduction

Several authors have given definitions of an end point of a continuum, making use of properties which are possessed by an end point of a straight line interval, but which are not possessed by any interior point of the interval. We shall show in the present paper that the properties used in these so-called "definitions" are not logically equivalent, and shall determine the logical relations which do exist among these properties under various conditions. In Part 2, we shall show the equivalence of a number of these properties in the case of a continuous curve. In Part 3 is shown the equivalence with the first set, of two additional properties in the case of a continuous curve of a special type. In Parts 4 and 5, we determine the logical relations existing among certain of these properties in the case of a bounded continuum. Finally, in Part 6, some theorems are proved concerning points having one or more of the given properties.

In this paper we shall consider only plane point sets, although in a number of cases it is obvious that our results are true in space of any number of dimensions.

In regard to the use of the word "end point," we intend hereafter to use this word only in the sense of a point of a continuous curve satisfying Wilder's definition or one of the other definitions equivalent to it, in other words, a point having any one of the properties 1–7 given in Part 2. We shall not use the word "end point" in referring to a point of a continuum which is not a continuous curve. The examples of Part 4 show that a point of a continuum may have certain of the given properties and yet be so placed with respect to that continuum as hardly to deserve the name of "end point."

2. Concerning properties 1–7 for a continuous curve

The object of this section is to prove the equivalence of the following

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† National Research Fellow in Mathematics.

properties of a point of a continuous curve.* In each case, P denotes a point of a continuous curve M.

Property 1. If PP' is any arc in M whose end points are P and any other point P' of M, then the set $M-(PP'-P)$ contains no connected subset consisting of more than one point which contains P.

Property 2. If PP' is any arc in M whose end points are P and any other point P' of M, then P is not a limit point of any connected subset of $M-PP'$.

Property 3. If N is any subcontinuum of M containing P, then the set $M-(N-P)$ contains no connected subset consisting of more than one point which contains P.

Property 4. P is not a cut point† of any subcontinuum of M.

Property 5. P is not contained in any subcontinuum of M which is irreducible‡ between two other points of M.

Property 6. If N is any subcontinuum of M containing P, then P is not a limit point of any connected subset of $M-N$.

Property 7. Given any positive number ϵ, there exists a domain containing P of diameter less than ϵ, whose boundary has just one point in common with M.

Property 1 is due to R. L. Wilder.§ Properties 2, 3, and 6 are modifications of property 1. Property 4 was suggested by Professor R. L. Moore. Property 5 is due to Yoneyama.‖ Property 7 is due to Menger.¶

G. T. Whyburn** has proved that in the case of a continuous curve, property 1 is equivalent to the following property: No arc in M contains P as an interior point. In the case of a continuum which is not a continuous

* For a number of equivalent definitions of a continuous curve, see R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 289–302. In the present paper we shall make one change in the definitions given in *Report*, i.e., we shall define a *continuum* as a closed and connected point set *containing more than one point.*

† If M is a connected point set, and P is a point of M, then if $M-P$ is not connected, P is said to be a *cut point* of M; if $M-P$ is connected P is said to be a *non-cut point* of M.

‡ A point set K is said to be *an irreducible continuum between two points A and B*, if K is a continuum and contains A and B, but contains no proper subset which is a continuum and contains A and B.

§ R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), pp. 340–377. See especially p. 358.

‖ K. Yoneyama, *Theory of continuous set* (sic) *of points*, Tôhoku Mathematical Journal, vol. 13 (1918), p. 130.

¶ K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 277–306.

** G. T. Whyburn, *Concerning continua in the plane*, these Transactions, vol. 29 (1927), pp. 369–400. See Theorem 12, p. 385.

curve, Whyburn uses property 6 as a definition of an end point of the continuum.

W. L. Ayres* has proved that in the case of a continuous curve, property 1 is equivalent to the following property: P is a non-cut point of M which belongs to no simple closed curve in M.

THEOREM 1. *If a point P of a continuous curve M has any one of the properties* 1–7, *it has all the others.*

In Part 4 of this paper, we shall show that if M is any bounded continuum, and P has property 7, it has property 6; if P has property 6, it has property 5; if P has property 5, it has property 4.

If P has property 4, it has property 2. For if it fails to have property 2, then M contains an arc PP', such that P is a limit point of a connected subset X of $M-PP'$. Let K denote the maximal connected subset of $M-PP'$ containing X. The point P can be joined to any point P'' of K by an arc† lying in K except for the point P, and therefore having only P in common with the arc PP'. The sum of the arcs PP' and PP'' is an arc $P'P''$ of which P is an interior point, and therefore a cut point. But this is contrary to the assumption that P has property 4.

If P has property 2, it has property 1. For if it fails to have property 1, then there is some arc PP' in M, such that $M-(PP'-P)$ contains a connected subset X containing P and such that $X-P$ is not vacuous. Let K be the maximal connected subset of $M-(PP'-P)$ containing P. Since K contains X, the set $K-P$ is not vacuous. If Q is any point of $K-P$, then K contains the maximal connected subset of $M-PP'$ containing Q. Let us denote this set by D. Since the point P has property 2, P is not a limit point of D, and therefore K contains other points besides P and points of D. The set D is closed, save for limit points on PP', and since the point P is not a limit point of D, no point of $K-D$ is a limit point of D. Also, no point of D can be a limit point of any set of points of M not in D‡, and therefore no point of D is a limit point of $K-D$. Therefore K is disconnected, which is contrary to our supposition concerning K.

If in the argument in the preceding paragraph, we replace the arc PP' by a subcontinuum N of M containing P, we can prove that if P has property 6, it has property 3. Obviously if P has property 3, it has property 6, and

* W. L. Ayres, *Concerning continuous curves and correspondences*, Annals of Mathematics, (2), vol. 28 (1927), pp. 396–418. See Theorem 3, p. 399.

† R. L. Wilder, loc. cit., Theorem 1, p. 342.

‡ R. L. Wilder, loc. cit. See the proof of Theorem 9, p. 360.

therefore properties 3 and 6 are equivalent. To complete the proof of Theorem 1, we need only show that if P has property 1, it has property 7.

If P has property 1, and PP' is any arc in M, then P is a limit point of a set of points L of $PP'-(P+P')$, such that if X is any point of L, then $PX-X$ and $P'X-X$ lie in different maximal connected subsets of $M-X$. For suppose this is not true, and the arc PP' contains a subarc PQ, such that for every interior point X of PQ, the sets $PX-X$ and $P'X-X$ lie in the same connected subset of $M-X$. Then, for each point X, there is an arc in $M-X$ joining P to Q, and this arc contains as a subset an arc AB, which has only its end points in common with PQ, and which is such that A is either Q or an interior point of the arc QX, and such that B is an interior point of the arc XP. Also, the point P is not a limit point of the collection of maximal connected subsets of $M-PQ$ that have limit points on XQ, because P is not a limit point of any one of them (since it has property 1), and only a finite number of these sets can be of diameter greater than half the distance from P to the nearest point of XQ.* Therefore there is a last point (necessarily different from P) on the arc XP to which an arc AB, of the type described above, can be constructed. We have therefore shown that corresponding to any point X, an arc AB as described above can be constructed having the additional property that no interior point of the arc BP can be joined to a point of $XQ-X$ by an arc having only its end points in common with PQ.

Let us then select a point B_0, which is an interior point of the arc PQ. Let A_1B_1 be an arc corresponding to B_0. The point A_1 is a point of B_0Q-B_0, and B_1 is an interior point of PB_0. Let A_2B_2 be an arc corresponding to B_1. The point A_2 is a point of $B_1B_0-B_1$, and B_2 is an interior point of PB_1. Continuing this process, for $n\geqq 2$, there exists an arc $A_{n+1}B_{n+1}$ corresponding to B_n, where A_{n+1} is a point of $B_nB_{n-1}-B_n$, and B_{n+1} is an interior point of PB_n.

Any set of points $B_0, B_1, B_2, \cdots$ on an arc PQ, and such that B_i follows B_{i+1}, must have a sequential limit point on PQ. We shall show that under the given conditions, this sequential limit point must be the point P. For suppose a sequence of this type has a point C different from P as a sequential limit point. Then there exists an arc $A'B'$ corresponding to C, where A' is a point of $CQ-C$, and B' is an interior point of PC. Some point of the sequence $B_0, B_1, B_2, \cdots$, say B_n, is an interior point of the arc CA'. By our method of constructing the arc $A_{n+1}B_{n+1}$ corresponding to B_n, no interior point of $B_{n+1}P$ can be joined to a point of B_nQ-B_n by an arc having only its end points in common with PQ. But B_{n+1} lies between C and A', and therefore

* W. L. Ayres, loc. cit., Theorem 1, p. 396.

B' is an interior point of $B_{n+1}P$, while A' is a point of B_nQ-B_n. The existence of the arc $A'B'$ shows that our given method of construction was not followed in constructing the arc $A_{n+1}\,B_{n+1}$. Having arrived at this contradiction by supposing C to be the limit of the sequence, it follows that P is the sequential limit point of any sequence $B_0, B_1, B_2, \cdots$, obtained as described above.

The continuum consisting of the arc PQ and the sequence of arcs A_1B_1, A_2B_2, $A_3B_3, \cdots$ is a continuous curve, every subcontinuum of which is a continuous curve.* Let M_1 denote the continuum consisting of P, the arcs $A_{2n+1}\,B_{2n+1}$ $(n=0, 1, 2, \cdots)$, and the arcs $B_{2n+1}\,A_{2n+3}$ $(n=0, 1, 2, \cdots)$ of the arc PQ. Let M_2 denote the continuum consisting of P, the arcs $A_{2n}\,B_{2n}$ $(n=1, 2, 3, \cdots)$, and the arcs $B_{2n}\,A_{2n+2}$ $(n=1, 2, 3, \cdots)$ of the arc PQ. These two continua have only the point P in common, and since each is a continuous curve, we can construct in each an arc having P as an end point. But in that case P fails to have property 1, which is contrary to hypothesis. We have thus established that if P has property 1, any arc PP' contains a set of cut points of M having P as a limit point, each of these points X being such that $PX-X$ and $P'X-X$ lie in different maximal connected subsets of $M-X$.

Let PP' be an arc in M, and let $P_1, P_2, P_3, \cdots$ be a sequence of points of PP' which are cut points of M of the type described above, and whose sequential limit point is P. Given any positive number ϵ_1, we can select a positive number ϵ, such that ϵ is less than ϵ_1, and is less than the distance from P to P'. The number of maximal connected subsets of $M-PP'$ of diameter greater than $\epsilon/6$ is finite, and since P is not a limit point of any one of these sets, we can select an integer n such that the diameter of PP_n is less than $\epsilon/6$, and such that no maximal connected subset of $M-PP'$ of diameter greater than $\epsilon/6$ has any limit points on PP_n. If we denote by N the maximal connected subset of $M-P_n$ that contains P, it follows that the diameter of N is less than $\epsilon/2$. The continuum $N+P_n$ cannot contain a simple closed curve enclosing the set $P_nP'-P_n$, otherwise the diameter of N would be greater than the distance from P to P', which is impossible.

If no simple closed curve in $M-N$ encloses a point of N, then if we add to $N+P_n$ all points of the plane which are interior to a simple closed curve in $N+P_n$, we obtain a continuum K which does not separate the plane. Let H_1 denote the continuum consisting of all points of $M+K-(K-P_n)$, and let H denote the continuum consisting of H_1 and all points of the plane which are interior to a simple closed curve in H_1. The two continua K and H

* H. M. Gehman, *Some conditions under which a continuum is a continuous curve*, Annals of Mathematics, (2), vol. 27 (1926), pp. 381–384. See especially Theorem 2, p. 382.

satisfy certain conditions* under which there exists a simple closed curve enclosing $K-P_n$ but not enclosing any other points of $K+H$, and containing P_n but not containing any other points of $K+H$. Therefore there exists a simple closed curve enclosing N, not enclosing $P_nP'-P_n$, and having only P_n in common with M.

In case a simple closed curve in $M-N$ encloses a point of N, it encloses all of N. Then, by Theorem 3 of S.P.S., there exists a simple closed curve having the properties mentioned at the end of the preceding paragraph.

Let us denote by J the simple closed curve enclosing N. In case J is of diameter less than ϵ, its interior is the domain required in order that P have property 7. In case J is of diameter greater than ϵ, we shall show how to replace J by a simple closed curve whose diameter is less than ϵ, which encloses N, and which has only P_n in common with M. Let us denote by I the interior of J.

Let us denote by K, the continuum consisting of all points of M in $J+I$, and all points of the plane which are interior to a simple closed curve of M in $J+I$. Then by Theorem 1 of S.P.S., there exists a simple closed curve L which encloses K and is such that every point of L plus its interior is at a distance less than $\epsilon/4$ from some point of K. By the way in which the point P_n was selected, the diameter of K is less than $\epsilon/2$, and therefore the diameter of L is less than ϵ. Since the diameter of J is greater than ϵ, J is not entirely contained in L plus its interior, and therefore L contains some points which are interior to J. The two simple closed curves J and L satisfy the conditions† under which there exists a simple closed curve J' which is a subset of $J+L$, which contains an arc through P_n, and every point of whose interior is interior to both J and L. The simple closed curve J' has only P_n in common with M, because J' is a subset of J plus that portion of L which is interior to J, and this portion of L has no points in common with M. Furthermore, since every point of the interior of J' is a point of the interior of L, the diameter of J' cannot be greater than the diameter of L, that is, the diameter of J' is less than ϵ. The existence of the simple closed curve J' shows that P has property 7, as the interior of J' is the domain required in order that P have property 7. This completes the proof of Theorem 1.

DEFINITION. A point of a continuous curve which has properties 1–7 is said to be an *end point* of the continuous curve.

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* R. L. Moore, *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 469–476. See Theorem 2, p. 470. We shall refer to this paper hereafter as S.P.S.

† R. L. Moore, *On the Lie-Riemann-Helmholtz-Hilbert problem of the foundations of geometry*, American Journal of Mathematics, vol. 41 (1919), pp. 299–319. See especially Theorem 26, p. 311.

Note that in the course of the proof of Theorem 1, we have also proved the following theorem:

THEOREM 2. *An end point P of a continuous curve M has this property: given any positive number ϵ, there exists a simple closed curve enclosing P, of diameter less than ϵ, and having just one point in common with M. Conversely, if a point P of a continuous curve M has the above property, then P is an end point of M.*

3. CONCERNING PROPERTIES 8–9 FOR A CONTINUOUS CURVE

We shall now introduce two additional properties of a point P of a continuous curve M.

PROPERTY 8. If P' and P'' are any two points of M different from P, then any two subcontinua of M irreducible between P and P', and between P and P'', respectively, have in common a continuum containing P.

PROPERTY 9. There exists a positive number x, such that if P' and P'' are any two points of M at a distance less than x from P, then one of any two subcontinua of M irreducible between P and P', and between P and P'', respectively, contains the other.

Properties 8 and 9 are due to Yoneyama.* A property analogous to property 9 has also been given by Young.†

In Part 4 of this paper, it is shown that if M is any bounded continuum, and P has property 9, it has property 8; and if P has property 8, it has property 4. Therefore, in the case of a continuous curve, if P has property 8, it has properties 1-7. The following examples will show (1) that a point P of a continuous curve M may have properties 1-7, without having properties 8 or 9; (2) that a point P of a continuous curve M may have properties 1-8, without having property 9.

EXAMPLE 1. Let M consist of the point $(0, 0)$, and the circles with center at $(3/2^n, 0)$ and with radius equal to $1/2^n$, for $n=1, 2, 3, \cdots$, and let P be the point $(0, 0)$. The point P has properties 1–7, but not properties 8 and 9.

EXAMPLE 2. Let M consist of the straight line intervals between $(0, 0)$ and $(1, 0)$, and between $(1/2^n, 0)$ and $(1/2^n, 1/2^n)$, for $n=1, 2, 3, \cdots$, and let P be the point $(0, 0)$. The point P has properties 1-8, but not property 9.

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* K. Yoneyama, *On continuous set* (sic) *of points*, II, Tôhoku Mathematical Journal, vol. 18 (1920), p. 254.

† W. H. Young and G. C. Young, *The Theory of Sets of Points*, 1906, p. 220.

THEOREM 3. *A necessary and sufficient condition that a point P of a continuous curve M have property* 8, *is that P have properties* 1-7, *and that any arc PQ of M contains a sub-arc PX, every interior point of which is a cut point of M.*

Let P be an end point of a continuous curve M such that if PQ is any arc of M, then PQ contains a sub-arc PX every interior point of which is a cut point of M. This is equivalent to the statement that if PQ is any arc of M, the point P is not a limit point of those maximal connected subsets of $M-PQ$ that have more than one limit point on PQ. We shall now show that P also has property 8.

Let us select a definite point Q and a definite arc PQ. Let ϵ be a positive number which is less than the distance from P to Q, and less than the distance from P to any point of a maximal connected subset of $M-PQ$ that has more than one limit point on PQ. By Theorem 2, we can construct a simple closed curve J enclosing P, of diameter less than ϵ, and having just one point X in common with M. The point X is necessarily an interior point of the arc PQ.

If P' is any point exterior to J, any subcontinuum of M which is irreducible between P and P' must contain X, and therefore can be expressed as the sum of two continua irreducible between P' and X and between X and P, respectively, and having only X in common. We shall now show that any subcontinuum of M irreducible between P and any point Y lying in J plus its interior, has an arc in common with the arc PX of the arc PQ.

Note that under our given conditions, every interior point of the arc PX is a cut point of M. Therefore if Y is any point of $PX-P$, any connected subset of M containing Y and P necessarily contains all points of the arc PY, and therefore any subcontinuum of M irreducible between P and Y must coincide with this arc. If Y is a point of a maximal connected subset of $M-PQ$ in the interior of J, any connected subset of M containing Y and P must contain the point Z which is the limit point on PQ of the maximal connected subset of $M-PQ$ that contains Y. Since P has properties 1-7, Z is different from P. As before, any connected subset of M that contains Z and P must contain the arc PZ, and therefore any subcontinuum of M irreducible between P and Y has thc arc PZ in common with PQ.

Therefore if P' and P'' are any two points of M, each one of any two subcontinua of M irreducible between P and P', and between P and P'' respectively, has an arc containing P in common with PQ, and the common part of these two arcs is the continuum required in order that P have property 8. Therefore the condition is sufficient.

Suppose now that a continuous curve M contains a point P which has properties 1-8, and suppose that for some arc PQ, the point P is a limit point of non-cut points of M on PQ, and is therefore also a limit point of maximal connected subsets of $M-PQ$ which have more than one limit point on PQ. Let us select a sequence $D_1, D_2, D_3, \cdots$ of maximal connected subsets of $M-PQ$ having P as a limit point, such that D_i has at least two limit points on PQ, and such that every limit point of D_{i+1} on PQ lies between P and each limit point of D_i on PQ. The sequence can be selected so as to satisfy this latter condition, because P is not a limit point of any one of the sets D_i (because P has properties 1-8), and because the number of maximal connected subsets of $M-PQ$ of diameter greater than any given positive number is finite.

If A_i and B_i are two points of PQ which are limit points of D_i, an arc can be constructed from A_i to B_i in the set $D_i+A_i+B_i$. The set of arcs A_iB_i $(i=1, 2, 3, \cdots)$ plus the set of arcs B_iA_{i+1} $(i=1, 2, 3, \cdots)$ of the arc PQ, plus the point P, is a continuous curve, by the argument given in the proof of Theorem 1, and therefore this set contains an arc from A_1 to P. However this arc A_1P and the arc PQ do not have in common a continuum containing P, and therefore P fails to have property 8, which is contrary to hypothesis. Therefore the condition is necessary.

COROLLARY 3a. *If a point P of an acyclic* continuous curve has any one of the properties* 1-8, *it has all the others.*

THEOREM 4. *A necessary and sufficient condition that a point P of a continuous curve M have property* 9, *is that if PQ is any arc of M, then P is not a limit point of $M-PQ$.*

Let P be a point having the given property, and let us select a definite arc PQ of M. Then since P is not a limit point of $M-PQ$, there exists a point Y of PQ, such that no point of the arc PY is a limit point of $M-PQ$. Let x be a positive number which is less than the distance from P to any point of $M-PY$. If P' is any point of M at a distance less than x from P, the point P' is a point of the arc PY. Any connected subset of M containing both P and P' must contain the arc PP', and therefore the only subcontinuum of M which is irreducible between P and P' is the arc PP'. Therefore if P' and P'' are any two points of M at a distance less than x from P, one of the two arcs PP', PP'' will contain the other, and therefore P has property 9, and the condition is sufficient.

* An *acyclic continuous curve* is a continuous curve containing no simple closed curve.

The condition is necessary, for suppose P has property 9 (and therefore properties 1-8), and yet there is an arc PQ of M such that P is a limit point of $M-PQ$. Then if we select any positive number x, there are points of two different maximal connected subsets of $M-PQ$ at a distance less than x from P, for if there were only one, P would fail to have properties 1-9. Let these sets be D_1 and D_2. If P_i is a point of D_i, an arc P_iP can be constructed in D_i+PQ and evidently neither of the arcs P_1P, P_2P contains the other, contrary to our hypothesis that P has property 9.

Corollary 4a. *If a point P of an arc has any one of the properties* 1-9, *it has all the others.*

To recapitulate:

Theorem 5. *For a point of a continuous curve, properties* 1-7, *Whyburn's property, Ayres' property, and the property mentioned in Theorem* 2 *are equivalent; property* 8 *is stronger than any of these; and property* 9 *is stronger than property* 8. *For a point of an acyclic continuous curve, properties* 1-8 *are equivalent, and property* 9 *is stronger than any of them. For a point of an arc, properties* 1-9 *are equivalent.*

4. Concerning properties 4-9 for a bounded continuum

In this part, we shall consider the logical relations between properties 4-9, for the case where M is bounded continuum.

Theorem 6. *If a point P of a bounded continuum M has property* 9, *it has property* 8.

Suppose P has property 9, but not property 8. Then there exist two points P' and P'' of M, and two subcontinua N', N'' of M irreducible between P and P', and between P and P'', respectively, but which do not have in common any continuum containing P. Since P has property 9, we can select a point Q' of N', and a point Q'' of N'' sufficiently close to P that one of any two subcontinua of M irreducible between P and Q', and between P and Q'', contains the other. Since Q' and P are points of N', the set N' contains a subcontinuum K' which is irreducible between Q' and P, and similarly N'' contains a subcontinuum K'' which is irreducible between Q'' and P. We have shown above that one of the two continua K', K'' contains the other. Suppose K' contains K''. If so, the two continua N' and N'' have in common the continuum K'' which contains P, which is contrary to our supposition that N' and N'' do not have in common a continuum containing P.

THEOREM 7. *If a point P of a bounded continuum M has property* 8, *it has property* 4.

Suppose P has property 8, but not property 4. Then, M contains a subcontinuum N containing P, such that $N-P$ is not connected. Let H_1+H_2 be any method of expressing M_1-P as the sum of two sets having no points in common and neither containing a limit point of the other. Then H_1+P and H_2+P are two subcontinua of M having only P in common. The set H_1+P contains a subcontinuum irreducible between P and any point P' of H_1+P, and H_2+P contains a subcontinuum irreducible between P and any point P'' of H_2+P. These two continua have only P in common, and therefore P does not have property 8, contrary to hypothesis.

THEOREM 8. *If a point P of a bounded continuum M has property* 7, *it has property* 6.

Suppose P has property 7, but not property 6. Then there is some subcontinuum N of M containing P, such that P is a limit point of some connected set L which is a subset of $M-N$. Let X be a point of $N-P$, and Y a point of L. Let ϵ be less than the distance from P to X, and less than the distance from P to Y. Since P has property 7, there exists a domain D which contains P, whose exterior contains both X and Y, and whose boundary has only one point Q in common with M. The connected set N contains the point P in D, and the point X exterior to D, and therefore contains a point of the boundary of D. Therefore Q is a point of N. But the connected set $L+P$ also contains the point P in D, and the point Y exterior to D, and therefore Q is also a point of $M-N$. But it is impossible for Q to be both a point of N and a point of $M-N$, and therefore if P has property 7, it also has property 6.

THEOREM 9. *If a point P of a bounded continuum M has property* 6, *it has property* 5.

Suppose P has property 6, but not property 5. Then M contains a subcontinuum N containing P, which is irreducible between two points P', P'' different from P. Let K be any subcontinuum of N containing P, but not P' or P'', and let D' be the maximal connected subset of $N-K$ that contains P'. Since P has property 6, P is not a limit point of D', and therefore if D' also contained P'', the continuum consisting of D' and its limit points in K would be a proper subcontinuum of N containing P' and P'', which is impossible. Therefore P'' lies in a maximal connected subset of $N-K$ different from D', and we shall denote the one in which P'' lies by D''. The set $K+D'+D''$ is a continuum which is a subset of N and contains P' and P'',

and must therefore be identical with N. Therefore $N-K$ consists of two and only two maximal connected subsets, one of which contains P', and the other P''. Let us denote by E' and E'' the sets of points of K which are limit points of D' and D'' respectively. Since P is not a point of either E' or E'', the sets E', E'' have no points in common. For if they had points in common, $D'+E'+D''+E''$ would be a proper subcontinuum of N containing both P' and P'', which is impossible. Also $D'+E'$ is irreducible between P' and any point of E', and $D''+E''$ is irreducible between P'' and any point of E'', while K is irreducible between some point of E' and some point of E''.

Let us select a sequence of subcontinua of N: $N_1, N_2, N_3, \cdots$, such that for $i=1, 2, 3, \cdots$, (1) N_i contains N_{i+1}, (2) N_i contains P, but not P' or P'', (3) the diameter of N_i is less than $1/i$. For each of the sets N_i, let D_i' and D_i'' denote the maximal connected subsets of $N-N_i$ containing P' and P'' respectively. The sequence of continua N_1+D_1', N_2+D_2', N_3+D_3', $\cdots$, has the property that each continuum contains the one following it in the sequence. There is therefore a continuum K' common to the members of the sequence. Evidently K' contains P' and P, but contains no points of any of the sets D_i''.

Also there is a continuum K'' common to all the members of the sequence N_1+D_1'', N_2+D_2'', N_3+D_3'', $\cdots$, and K'' contains P'' and P, but no points of any of the sets D_i'. Therefore the only points that K' and K'' can have in common are those points which are common to all members of the sequence $N_1, N_2, N_3, \cdots$, and the point P is the only point common to all the sets N_i. Therefore K' and K'' have only P in common.

The set K' can also be expressed as the point P plus the sum of the infinite collection of connected sets D_1', D_2', D_3', $\cdots$, each of which is contained in all that follow it in the sequence. It therefore follows that P is a non-cut point of K', and that therefore the point P, considered as a point of the continuum K'', is a limit point of the connected subset $K'-P$ of $M-K''$. That is, P fails to have property 6, which is contrary to hypothesis.

Theorem 10. *If a point P of a bounded continuum M has property* 5, *it has property* 4.

Suppose P has property 5, but not property 4. Then M contains a subcontinuum N, such that $N-P$ is disconnected. In that case, N can be expressed as the sum of two continua N_1 and N_2 having only P in common. Let P_i be a point of $N_i (i=1, 2)$, and let K_i be a subcontinuum of N_i which is irreducible between P_i and P. The continuum K_1+K_2 is irreducible between P_1 and P_2, and therefore P does not have property 5, which is contrary to hypothesis.

THEOREM 11. *If a point P of a bounded continuum M has both property 5 and property 9, it has property 6.*

We shall show that if P has properties 5 and 9, and if N is any subcontinuum of M containing P, then P is not a limit point of the set $M-N$, and therefore P has property 6. Suppose then that P has properties 5 and 9, but that for some subcontinuum N of M, the point P is a limit point of $M-N$. We shall show that this leads to a contradiction.

Let P_0 be a point of N whose distance from P is less than the distance x (of property 9). Then N contains a subcontinuum N_0 irreducible between P_0 and P, and P is also a limit point of the set $M-N_0$. If P' is any point of $M-N_0$ whose distance from P is less than x, every subcontinuum of M that contains P' and P must contain the continuum N_0, otherwise P fails to have property 9.

Let P_1 be a point of $M-N_0$ whose distance from P is less than x, and let N_1 be a subcontinuum of M irreducible between P_1 and P. We have just shown above that N_0 is a subset of N_1. Since N_0 contains P, the set N_1-N_0 is connected.

If N_1 contains all points of $M-N_0$ which are at a distance from P less than some constant k then if we add to N_1-N_0 the set of its limit points (which includes P), the resulting continuum must contain N_0, as we have shown above. In other words, N_0 is a continuum of condensation of N_1. But in that case N_1 is irreducible between P_1 and any point of N_0, and P fails to have property 5. Therefore there are points of $M-N_0$ arbitrarily close to P which are not points of N_1, that is, P is a limit point of the set $M-N_1$.

Therefore let us select a point P_2 of $M-N_1$ whose distance from P is less than $x/2$. If N_2 is any subcontinuum of M irreducible between P_2 and P, then N_2 contains N_1 and P is a limit point of the set $M-N_2$. In general, if (for $i=2, 3, 4, \cdots$) P_i is a point of $M-N_{i-1}$ whose distance from P is less than x/i, then any subcontinuum N_i of M which is irreducible between P_i and P, contains N_{i-1}, and P is a limit point of the set $M-N_i$.

Let us then select a definite sequence of points $P_0, P_1, P_2, \cdots$, and a definite sequence of continua $N_0, N_1, N_2, \cdots$ having the properties described above. We shall show that if K denotes the continuum consisting of $N_0+N_1+N_2+\cdots$ plus limit points, then every proper subcontinuum of K is a continuum of condensation of K and therefore K is indecomposable.*

* A continuum is said to be *indecomposable* if it cannot be expressed as the sum of two of its proper subcontinua.

Suppose some proper subcontinuum L of K is not a continuum of condensation of K. Evidently the points of L which are not limit points of $K-L$ do not lie entirely in the set of limit points of $N_0+N_1+N_2+\cdots$, and therefore L must contain a point Q of one of the sets $N_0, N_1, N_2, \cdots$, say N_j, such that Q is not a limit point of $K-L$.

Suppose L does not contain P. The closed set H consisting of $(K-L)$ plus limit points of $(K-L)$ is a proper subset of K, and consists of a collection (G) of maximal continua of H. The continuum G_1 of this collection that contains P can contain none of the points $P_j+P_{j+1}+P_{j+2}+\cdots$. The continuum L also can contain only a finite number of points of the sequence $P_0, P_1, P_2, \cdots$ and therefore there exists an integer k such that no point of the sequence $P_k, P_{k+1}, P_{k+2}, \cdots$ is a point of L or of G_1. Let P_m and P_n be two points such that $m\geqq k$, $n\geqq k$, and $m\geqq j$, $n\geqq j$. If P_m and P_n were in different continua of (G), then by adding to $L+G_1$ each of these continua in turn, we obtain two continua in which we can construct subcontinua of M irreducible between P_m and P, and between P_n and P respectively, neither of which contains the other, which is contrary to the hypothesis that P has property 9. Therefore all points of $P_{j+k}, P_{j+k+1}, P_{j+k+2}, \cdots$ are points of the same maximal continuum of H. Since P is a limit point of the set $P_{j+k}+P_{j+k+1}+P_{j+k+2}+\cdots$, it follows that P also belongs to the maximal continuum of H that contains this set. But this is impossible, as the continuum G_1 containing P contains no points of the set $P_j+P_{j+1}+P_{j+2}+\cdots$.

Suppose L contains P. If L contains any point P_n of the set $P_0+P_1+P_2+\cdots$, then L must contain all the points $P_0+P_1+\cdots+P_n$. If there were no integer k such that P_k is a point of $K-L$ then L would be identical with K, which is contrary to our supposition that L is a proper subcontinuum of K. Therefore there is some integer k, such that $P_k, P_{k+1}, P_{k+2}, \cdots$ are all points of $K-L$. As in the preceding paragraph, all the points $P_k, P_{k+1}, P_{k+2}, \cdots$ lie in the same continuum of (G), and this continuum also contains P. But this is impossible, as the continuum containing P cannot contain any points of the sequence $P_j+P_{j+1}+P_{j+2}+\cdots$. Therefore K is indecomposable.

Any indecomposable continuum K has this property: given any point P there are two other points such that K is irreducible between any two points of the three. It follows that if M contains an indecomposable continuum K containing P, then P does not have property 5. But this is contrary to hypothesis, and therefore Theorem 11 is true.

Note that the above argument establishes the following theorem:

THEOREM 12. *Given a sequence of points $P_0, P_1, P_2, \cdots$ whose sequential limit point is P, and a collection of continua $N_0, N_1, N_2, \cdots$, such that N_i is irreducible between P_i and P, and such that N_i is a proper subset of N_{i+1}. If the continuum K consisting of $N_0+N_1+N_2+\cdots$ plus limit points has the property that any subcontinuum of K irreducible between P_i and P is a subset of every subcontinuum of K irreducible between P_{i+1} and P, then K is an indecomposable continuum.*

This theorem can be used to establish the indecomposability of two examples due to Knaster.* To recapitulate:

THEOREM 13. *For a point of a bounded continuum,*
(1) *property* 9 *is stronger than property* 8;
(2) *property* 8 *is stronger than property* 4;
(3) *property* 7 *is stronger than property* 6;
(4) *property* 6 *is stronger than property* 5;
(5) *property* 5 *is stronger than property* 4;
(6) *properties* 5 *and* 9 *together are stronger than property* 6.

The following set of examples shows that no logical relations exist among properties 4-9, excepting those given in Theorems 6-11, and therefore the truth of Theorem 13 follows from the examples and from Theorems 6-11.

An end point of a straight line interval has all the properties 4-9; an interior point has none of them. The point P of example 1 (of Part 3) has properties 4-7, but not 8 or 9. The point P of example 2 has properties 4-8, but not 9.

EXAMPLE 3. Let M consist of the curve $y=\sin(1/x)$, for $0<x\leqq 1$, and the straight line interval from $(0, -1)$ to $(0, 1)$. Let P be the point $(0, 1)$, which has properties 4, 8, and 9, but not 5, 6, or 7.

EXAMPLE 4. Let M consist of the continuum of example 3, plus the curve $y=\sin(1/x)$, for $0>x\geqq -1$. Let P be the point $(0, 1)$, which has properties 4 and 8, but none of the properties 5-7, 9.

EXAMPLE 5. Let M consist of the continuum of example 3, plus the set of semicircles lying on the negative side of the y-axis, with center at the point $(0, 1-3/2^n)$ and radius equal to $1/2^n$, for $n=1, 2, 3, \cdots$. Let P be the point $(0, 1)$, which has property 4, but none of the properties 5-9.

EXAMPLE 6. Let M consist of the continuum of example 2, plus the set of curves

* Contained in C. Kuratowski, *Théorie des continus irréductibles entre deux points*, Fundamenta Mathematicae, vol. 3 (1922), pp. 200–223. These examples are described on pp. 209–210 and on pp. 216–217 respectively.

$$y = \frac{1}{2^{n+1}} + \left(\frac{1}{2^n} - \frac{x}{2}\right) \sin\left(\frac{1}{x - 1/2^n}\right),$$

between $(1/2^n) < x \leqq (1/2^{n-1})$, for $n = 1, 2, 3, \cdots$. Let P be the point (0, 0), which has properties 4 and 5, but none of the properties 6-9.

EXAMPLE 7. Let M consist of the straight line intervals from (1, 0) to (0, 0), and from (1, 0) to $(0, 1/n)$, for $n = 1, 2, 3, \cdots$. Let P be the point (0, 0), which has properties 4, 5, and 8, but none of the properties 6, 7, 9.

EXAMPLE 8.* Let M consist of the straight line intervals from (0, 0) to (1, 0), from $(1/2^n, 0)$ to $(1/2^n, 1/2^n)$, from $(1/2^n, 1/2^n)$ to $(-1/2^n, 1/2^n)$, from $(-1/2^n, 1/2^n)$ to $(-1/2^n, -1/2^n)$, from $(-1/2^n, -1/2^n)$ to $(1, -1/2^n)$, from $(1, -1/2^n)$ to $(1, -3/2^{n+2})$, from $(1, -3/2^{n+2})$ to $(0, -3/2^{n+2})$ for $n = 0, 1, 2, \cdots$. Let P be the point (0, 0), which has properties 4, 5, 6, and 8, but not property 7 or property 9.

EXAMPLE 9. Let M consist of the continuum of example 8, plus the set of semicircles lying on the positive side of the x-axis, with center at the point $(3/2^n, 0)$ and radius equal to $1/2^n$, for $n = 2, 3, 4, \cdots$. Let P be the point (0, 0), which has properties 4, 5, 6, but none of the properties 7, 8, and 9.

EXAMPLE 10. Let M_0 denote the indecomposable continuum described by Knaster in his thesis,† and designated there by K_1. This continuum lies within the square whose vertices are (0, 0), (1, 0), (1, 1), (1, 0) and is irreducible between the point (1, 0) and any point which cannot be joined to (1, 0) by an arc in M_0. The continuum M_0 can also be constructed by the method used in constructing the second example in Kuratowski's article.

Let $M_n (n = 1, 2, 3, \cdots)$ be the set of points (x', y') obtained by subjecting the points (x, y) of M_0 to the following transformation: $x' = (x + 2^{n+1} - 2)/2^n$, $y' = y/2^n$. Let P be the point (2, 0), and let M be the continuum consisting of $P + M_0 + M_1 + M_2 + \cdots$. The point P has properties 4, 5, 6, 8, and 9, but not property 7.

EXAMPLE 11. Let M denote an indecomposable continuum every subcontinuum of which is indecomposable. An example of such a continuum has been given by Knaster,‡ who designates it by K_3. If P is *any* point of M, P has properties 4, 8, and 9, but none of the properties 5-7. This combination of properties of a point has already been illustrated in example 3.

5. CONCERNING PROPERTY 3 FOR A BOUNDED CONTINUUM

The following theorem follows from the definitions of properties 3 and 6.

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* This example is due to Whyburn, loc. cit., proof of Theorem 30.

† B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fundamenta Mathematicae, vol. 3 (1922), pp. 245–286. This example is described on pp. 269–271.

‡ Loc. cit., pp. 275–279.

THEOREM 14. *If a point P of a bounded continuum M has property* 3, *it has property* 6.

We have not been able to determine whether or not P must have property 3 when it has property 6. However, the arguments given to prove Theorems 8 and 11 also establish the following theorems concerning property 3.

THEOREM 15. *If a point P of a bounded continuum M has property* 7, *it has property* 3.

THEOREM 16. *If a point P of a bounded continuum M has property* 5 *and property* 9, *it has property* 3.

We can combine Theorem 13 and the results of this section into the following theorem.

THEOREM 17. *For a point of a bounded continuum*
(1) *property* 9 *is stronger than property* 8;
(2) *property* 8 *is stronger than property* 4;
(3) *property* 7 *is stronger than property* 3;
(4) *property* 6 *is stronger than property* 5;
(5) *property* 5 *is stronger than property* 4;
(6) *properties* 5 *and* 9 *are stronger than property* 3;
(7) *either properties* 3 *and* 6 *are equivalent, or property* 3 *is stronger than property* 6.

In case property 3 is stronger than property 6, the question remains whether examples exist of a point having properties 4-6, 8 but not 3, 7, 9, and of a point having properties 4-6, but not properties 3, 7-9, or whether some further logical relations exist among properties 3-9 for a bounded continuum.

6. SOME THEOREMS CONCERNING POINTS WITH PROPERTIES 1-9

In the preceding discussion we have assumed the following theorem, whose truth follows from the definitions of properties 1-9.

THEOREM 18. *If a point P of a bounded continuum M has property x (where $x=1, 2, \cdots, 9$), and N is a subcontinuum of M containing P, then P has property x in the continuum N.*

THEOREM 19. *The set of points of a bounded continuum which have property 3 is totally disconnected.*

If the theorem were not true, there would exist a bounded continuum M which contains a connected subset K consisting of more than one point, such that every point of K has property 3. Let P be a point of K.

If K is a proper subset of M, and Q is any point of $M-K$, any subcontinuum N of M which is irreducible between P and Q contains no points of K, except P, because $N-P-Q$ can contain no points having property 5, and therefore none having property 3, by Theorems 14 and 9. Therefore $M-(N-P)$ contains K, and P fails to have property 3, which is contrary to hypothesis.

If K is identical with M, any subcontinuum N of M which is irreducible between any two points P and Q of M, consists entirely of points of K. But no point of $N-P-Q$ can have property 5, and therefore none can have property 3. Therefore no point of $N-P-Q$ is a point of K, which is contrary to hypothesis. Therefore Theorem 19 is true.

COROLLARY 19a. *The set of points of a bounded continuum which have property* 7, *is totally disconnected.**

COROLLARY 19b. *The set of points of a bounded continuum which have both properties* 5 *and* 9 *is totally disconnected.*

COROLLARY 19c. *The set of end points of a continuous curve is totally disconnected.*†

THEOREM 20. *A bounded continuum which is irreducible between two of its points, cannot contain more than two points with property* 5.‡

THEOREM 21. *The set of points of a bounded continuum which have property* 5, *contains no continuum.*

COROLLARY 21a. *The set of points of a bounded continuum which have property* 6, *contains no continuum.*

However, the set of points of a bounded continuum which have property 9 may contain a continuum, as was shown in example 11. Similarly with the set of points having property 8, and with the set having property 4.

THEOREM 22. *If M is a bounded continuum, and K is any subset of the set of points of M which have property* 5, *then $M-K$ is strongly connected.*

* Menger, loc. cit., p. 283, Theorem V.

† Whyburn, loc. cit., Theorem 21. See footnote on p. 391. We have shown in Part 2 of this paper that an end point in Whyburn's sense is equivalent to an end point in Menger's sense, in the case of a continuous curve.

‡ In Whyburn's Theorem 31 the second paragraph should be omitted, as his "end point" has property 5, and therefore P cannot be any point other than A or B.

If this is not true, then $M-K$ contains two points P and Q such that every subcontinuum of M containing P and Q contains a point of K. In particular, any subcontinuum N of M irreducible between P and Q contains a point of K. But no point of $N-P-Q$ can have property 5, and therefore no point $N-P-Q$ is a point of K, which is a contradiction.

COROLLARY 22a. *If M is a continuous curve, and K is any subset of the end points of M, then $M-K$ is strongly connected.**

By example 11, we see that Theorem 22 is not true if property 5 is replaced by any of the properties 4, 8, or 9. In fact, in that case $M-K$ may be disconnected. The theorem remains true, of course, if property 5 is replaced by any of the properties 3, 6, or 7.

THEOREM 23. *A necessary and sufficient condition that a bounded continuum M be an acyclic continuous curve, is that every non-cut point have property* 5.

The necessity of the condition follows from the fact that every non-cut point of an acyclic continuous curve is an end point,† and therefore has property 5.

The condition is sufficient, because if every non-cut point of a bounded continuum M has property 5, the set of non-cut points of M is identical with the set of points of M which have property 5. If K is any subcontinuum of M, then K contains a subcontinuum N which is irreducible between two points X and Y, and therefore no point of $N-X-Y$ has property 5. In other words, every point of $N-X-Y$ is a cut point of M. But if every subcontinuum K of M contains uncountably many cut points of M, then M is an acyclic continuous curve.‡

The sufficiency of the condition can also be established by Theorem 22 and Whyburn's Theorem 32.

Note that the condition remains necessary and sufficient if we replace property 5 by any of the stronger conditions 3, 6, or 7. In case property 5 is replaced by property 4 or property 8 the condition is necessary, but not sufficient, as Example 11 shows. If property 5 is replaced by property 9, the condition is neither necessary nor sufficient.

THEOREM 24. *If a point P of a bounded continuum M has property* 7, *then P is a limit point of cut points of M.*

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* W. L. Ayres, loc. cit., Theorem 6, p. 401.

† Wilder, loc. cit., Theorem 7, p. 358.

‡ R. L. Moore, *Concerning the cut-points of continuous curves*, etc., Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106. See Theorem *E*, p. 103.

Given any positive number ϵ, there is a cut point of M whose distance from P is less than ϵ and therefore P is a limit point of cut points of M. Since a point having any of the properties 3-9 is a non-cut point of M, a point having property 7 is a non-cut point of M which is a limit point of cut points. However, the converse is not true, as a point P may be a non-cut point and a limit point of cut points, and still not have any of the properties 3-9, even if M is a continuous curve. An example which shows this, is a continuous curve M consisting of the continuum of example 2, plus the straight line intervals from (0, 0) to (0, 1), from (0, 1) to (1, 1), and from (1, 1) to (1, 0), where P is the point (0, 0).

The above theorem is not true if we replace property 7 by any or all of the properties 3-6, 8-9, as is shown by example 10, where P has all 3-6, 8-9, but is not a limit point of cut points of M.

COROLLARY 24a. *If P is an end point of a continuous curve M, then P is a limit point of cut points of M.*

We have already proved Corollary 24a incidentally in the proof of Theorem 1, where we used this fact to establish that if P has property 1, it has property 7.

THEOREM 25. *If a point P of a bounded continuum M has property* 7, *then M is connected im kleinen at P.*

Given any positive number ϵ, there is a domain D containing P of diameter less than ϵ, and such that the set of points of M in D plus its boundary is a continuum N. If δ is any positive number which is less than the distance from P to any point of the exterior of D, then any two points of M at a distance less than δ from P are points of N, and since N is of diameter less than ϵ, it follows that M is connected im kleinen at P.

THEOREM 26. *If a point P of a bounded continuum M has property* 5 *and property* 9, *then M is connected im kleinen at P.*

In the proof of Theorem 11, we showed that if P has properties 5 and 9, and if N is any subcontinuum of M containing P, then P is not a limit point of the set $M-N$. Given any positive number ϵ, let N be a subcontinuum of M of diameter less than ϵ and containing P, and let δ be less than the distance from P to any point of $M-N$. Then as in the proof of Theorem 25, it follows that M is connected im kleinen at P.

These two theorems might be combined into the single theorem that M is connected im kleinen at P if M has properties 3-7, or 3-6, 8, and 9. Example 11 shows that M is not necessarily connected im kleinen at P, if P has proper-

ties 4, 8 and 9, and example 8 shows that P may have properties 3-6, and 8, and still M need not be connected im kleinen at P. Therefore the hypothesis of neither of the above theorems can be weakened.

Whyburn* has shown that if a point P of a bounded continuum M has property 6 and is accessible from some point of a domain complementary to M, then M is connected im kleinen at P. This naturally raises the question as to the conditions under which a point P having certain of the given properties is accessible from a domain complementary to M.

If M is a continuous curve, every point of M (whether it be an end point or not) is accessible from every complementary domain on whose boundary the point lies. However, a point may be an end point of a continuous curve and also have property 8, and still not be on the boundary of any complementary domain. An example which shows this, is the continuous curve M consisting of the straight line interval from (0, 0) to (1, 0), and the set of circles $x^2+y^2=1/n^2$, for $n=1, 2, 3, \cdots$. The point $P=(0, 0)$ has properties 1-8, but is not on the boundary of any complementary domain.

However, if an end point of a continuous curve has property 9, it is a boundary point of a complementary domain, by Theorem 4.

If M is a bounded continuum (not necessarily a continuous curve), a point P may have all the properties 3-9, and still not be a boundary point of a complementary domain. An example which shows this, is the continuum consisting of the point (0, 0), the set of circles $x^2+y^2=1/4^n$, and the two sets of curves whose equations in polar coördinates are $r=(3+\theta/(\theta+1))/2^{n+2}$, for $\theta\geqq 0$, and $r=(3-\theta/(\theta+1))/2^{n+2}$, for $\theta\geqq 0$, and for $n=0, 1, 2, \cdots$. The point $P=(0, 0)$ has properties 3-9, but is not on the boundary of any domain complementary to M. Therefore, in the two following theorems, it is necessary to assume that P is on the boundary of a complementary domain.

THEOREM 27. *If a point P of a bounded continuum M has property* 7 *and is on the boundary of a domain D complementary to M, then P is accessible from D.*

Since P has property 7, if C is any circle about P as center, there exists a domain E interior to C, containing P, whose exterior contains points of D, and whose boundary F has just one point X in common with M. The domain E can also be selected so that $F-X$ is connected.† The domain E contains

* Loc. cit., Theorem 30, p. 395.

† R. L. Moore, *Concerning the sum of a countable number of mutually exclusive continua in the plane*, Fundamenta Mathematicae, vol. 6 (1924), p. 191, Theorem 3.

points of D, and since the exterior of E also contains points of D, it follows that all points of $F-X$ are points of D. If C_1 is any circle about P as center, and interior to E, any two points P_1 and P_2 of D interior to C_1 can be joined by a connected subset of D interior to C,—the connected subset consisting of $F-X$ plus the points in E of any arc joining P_1 to P_2 in D. Hence D is "connected near P" and hence* P is accessible from D.

THEOREM 28. *If a point P of a bounded continuum M has property* 5 *and property* 9, *and is on the boundary of a domain D complementary to M, then P is accessible from D.*

If B is the boundary of D, then P has properties 3-6, 8-9 in the continuum B, by Theorem 18. Therefore $B-P$ is connected and B contains no continuum of condensation containing P, and therefore D is connected near P, and P is accessible from D.†

Again, we might combine the above two theorems into the single theorem that P is accessible from a complementary domain D, if P is on the boundary of D, and has either properties 3-7, or properties 3-6, 8-9. Examples 8 and 11 show that neither hypothesis can be weakened and the theorem remain true, as these examples show that a point may have properties 4, 8, and 9 or properties 3-6, and 8, and be on the boundary of a complementary domain of M, and still not be accessible from that domain.

In conclusion, I wish to express my sincere appreciation to Professor R. L. Moore for his inspiring assistance in the preparation of all the papers written during my year as National Research Fellow.

* R. G. Lubben, *Concerning connectedness near a point set.*

† R. G. Lubben, loc. cit.

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THE APPORTIONMENT OF REPRESENTATIVES IN CONGRESS*

BY

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Introduction

In the absence of any provision for fractional representation in Congress, the constitutional requirement that the number of representatives of each state shall be proportional to the population of that state cannot be carried out exactly; some deviation from strict proportionality is unavoidable, on account of the necessary adjustment of fractions.

Thus, between any two states, there will practically always be a certain inequality which gives one of the states a slight advantage over the other. A transfer of one representative from the more favored state to the less favored state will ordinarily reverse the *sign* of this inequality, so that the more favored state now becomes the less favored, and vice versa. Whether such a transfer should be made or not depends on whether the *amount* of inequality between the two states after the transfer is less or greater than it was before; if the "amount of inequality" is reduced by the transfer, it is obvious that the transfer should be made.

The fundamental question therefore at once presents itself, as to how the "*amount of inequality*" between two states is to be measured. This is a mathematical question of quite unexpected complexity, which has been discussed on a scientific basis only within the last few years. The best solution of the problem appears to be the *Method of Equal Proportions*, which it is the purpose of the present paper to explain.†

* Presented to the Society, December 28, 1920, February 26, April 23, September 8, and December 28, 1921, and February 25, 1922; with the subsequent addition of a number of new examples and tables; and read, in part, before the Mathematical Association of America, December 31, 1926; received by the editors in January, 1927.

† See E. V. Huntington, *A new method of apportionment of representatives*, Quarterly Publication of the American Statistical Association, September, 1921, pp. 859–870; also the *Report upon the Apportionment of Representatives*, prepared by the Joint Committee of the American Statistical Association and the American Economic Association to Advise the Director of the Census, and published in the same journal, December, 1921, pp. 1004–1013. This Report, which pronounces in favor of the Method of Equal Proportions, is reprinted in full in Hon. E. W. Gibson's Remarks in the Congressional Record for April 7, 1926, pp. 6840–6842. The Method of Equal Proportions was incorporated in the Bill (H.R. 17378) introduced by Mr. Fenn in the House of Representatives, March 2, 1927; see the Report of Hearings held in January and February, 1927, before the House Committee on the Census (69th Congress, 2d Session), and the Congressional Record for March 2, 1927, pp. 5323–5331.

A first basis for the Method of Equal Proportions

The first measure of the "amount of inequality" between two states, which suggests itself, is based on the size of the "congressional district," that is, the result of dividing the population of the state by the number of its representatives.

For example, if the population of a certain state A is $A = 1{,}000{,}000$, and the number of its representatives is $a=4$, then the size of a congressional district in that state will be $A/a=250{,}000$. If the population of a second state B is B and the number of its representatives is b, then the size of the congressional district in the second state is B/b.

Now in a perfect apportionment, these two numbers would be exactly equal:

$$A/a = B/b\,;$$

hence, in any practical case, the inequality between these two numbers—that is, the inequality between the two congressional districts, A/a and B/b—may be taken as a measure of the "amount of inequality" between the two states A and B. If this inequality can be reduced by a transfer of a representative from one state to the other, then, according to this first criterion, the transfer should be made.

The rather vague concept of the inequality between two states is thus reduced to the more definite concept of the inequality between two numbers.

The question then comes down to this: what shall be meant by the inequality between these two numbers? Shall we mean the *absolute difference* between the two numbers, or the *relative difference* between them? If the size of the congressional districts is large, say 250,000 in one state and 250,005 in the other, then the difference of five people is of little consequence in so large a number. But if the districts were themselves very small, say 10 and 15, then the same difference of five people becomes important; 15, we say, is larger than 10 by fifty per cent, while 250,005 is larger than 250,000 by only (1/500)th of 1 per cent.

In the present problem it is clearly the relative or percentage difference, rather than the mere absolute difference, which is significant.* Our first criterion for a good apportionment may therefore be precisely formulated as follows:

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* The relative or percentage difference between two numbers is here thought of as the absolute difference divided by the smaller number. For the present purpose it might equally well be thought of as the absolute difference divided by the larger number, or the absolute difference divided by the (arithmetic, geometric, or harmonic) mean between the two numbers.

TEST 1. *If the relative difference between the congressional districts, A/a and B/b, belonging to any two states can be reduced by a transfer of a representative from one state to the other, then this transfer should be made.*

One further question remains. It is not obvious that a transfer which improves the situation between one pair of states, A and B, may not make the situation worse between one of these states and some other state; in other words, it is not obvious that the test can be applied to all pairs of states simultaneously.

It will be shown below, however, that this is a "workable" test; that is, by successive applications of the test, it is always possible to arrive at a final apportionment which cannot be "improved" by any further transfer between any two states.

The only known method of apportionment which satisfies Test 1 proves to be the Method of Equal Proportions.

A SECOND BASIS FOR THE METHOD OF EQUAL PROPORTIONS

A second, and equally obvious, method of defining the "amount of inequality" between two states is based, not on the ratio A/a, but on the ratio a/A (that is, the number of representatives divided by the population of the state). This number a/A is a small fraction which can be interpreted as the *individual share of a representative* which each inhabitant in the given state may be said to control.

For example, if the number of inhabitants in a given state is $A=1{,}000{,}000$, and the number of representatives is $a=4$, then the "individual share" of a representative which each inhabitant of that state can claim is $a/A=1/250{,}000=0.000004$. If the population of a second state is B and the number of its representatives is b, then the "individual share" in the second state is b/B.

Now in a perfect apportionment, these two numbers would be exactly equal:

$$a/A = b/B\ ;$$

hence, in any practical case, the inequality between these two numbers—that is, the inequality between the individual shares, a/A and b/B—may be taken as the measure of the "amount of inequality" between the two states A and B; and here, as before, it is clearly the relative or percentage difference, rather than the mere absolute difference, which is significant.

Our second criterion for a good apportionment may therefore be precisely formulated as follows:

TEST 2. *If the relative difference between the two "individual shares," a/A and b/B, belonging to any two states, can be reduced by a transfer of a representative from one state to the other, then this transfer should be made.*

Here again it will be shown that this is a "workable" test; and the only known method of apportionment which satisfies this Test 2 is the same Method of Equal Proportions which also satisfies Test 1.

WORKING RULE FOR THE METHOD OF EQUAL PROPORTIONS

We now turn to a purely technical question, of little interest except to the computers in the Bureau of the Census.

Given, the populations of the several states; and given, the size of the House, that is, the total number of representatives to be assigned; how shall we actually compute an apportionment which will satisfy Test 1 and Test 2? The practical working rule for the computation is as follows:

First, assign one representative to each state (here 48 in number).

Next, for each state, make out a series of cards, each card containing: (1) the name of the state; (2) a serial number, k, starting with 2 and running up to a number somewhat greater than the number of representatives that that state is expected to receive; and (3) a "rank index," found by multiplying the population of the state by a certain "multiplier," given, for each serial number, in the adjoining table.

Method EP	
No.	Multiplier
2	$1/[(1\cdot 2)]^{1/2}$
3	$1/[(2\cdot 3)]^{1/2}$
4	$1/[(3\cdot 4)]^{1/2}$
..	....

Then combine all these series of cards into a single series, arranged in order of the "rank indices," from the highest to the lowest, thus forming what may be called a "priority list," for the given populations, and any size of House.*

Finally, assign additional representatives (after the first) to the several states in the order in which the cards occur in this "priority list," continuing the assignment as far as may be necessary to fill up a House of any desired size.

An apportionment worked out according to this rule will always satisfy Test 1 and Test 2, as will be shown below. In practice, it may be found convenient to number the cards of the "priority list" consecutively, in red ink, beginning with the number (here 49) one greater than the number of states, and continuing until any desired total number of representatives

* In case two cards bear the same index number, the state having the larger population may be given priority. This case of a "tie" will be extremely rare, however, on account of the irrationality of the "multipliers" (see a later paragraph).

(say 435) has been reached. For most purposes, however, the earlier part of the list may be omitted.

In the following table the multipliers are given to seven decimal places.

Table of Multipliers (Method EP)

No.	Multiplier	No.	Multiplier	No.	Multiplier	No.	Multiplier
1	....	14	.074 1249	27	.037 7426	40	.025 3185−
2	.707 1068	15	.069 0066	28	.036 3696	41	.024 6932
3	.408 2483	16	.064 5497	29	.035 0931	42	.024 0981
4	.288 6751	17	.060 6339	30	.033 9032	43	.023 5310
5	.223 6068	18	.057 1662	31	.032 7913	44	.022 9900
6	.182 5742	19	.054 0738	32	.031 7500+	45	.022 4733
7	.154 3033	20	.051 2989	33	.030 7729	46	.021 9793
8	.133 6306	21	.048 7950+	34	.029 8541	47	.021 5066
9	.117 8511	22	.046 5242	35	.028 9886	48	.021 0538
10	.105 4093	23	.044 4554	36	.028 1718	49	.020 6197
11	.095 3463	24	.042 5628	37	.027 3998	50	.020 2031
12	.087 0388	25	.040 8248	38	.026 6690	51	.019 8030
13	.080 0641	26	.039 2232	39	.025 9762		

In this table, if k = the serial number, the "multiplier" $= 1/[(k-1)k]^{1/2}$. The entries in the table may be verified, with a computing machine, by the process of squaring and taking reciprocals, without extracting square roots.

An illustration of the use which may be made of the table of multipliers, even before the priority list is completed, is the following: Any state A will receive its 43d representative before another State B receives its 8th representative, provided the population of State A multiplied by 0.0235310 is greater than the population of State B multiplied by 0.1336306.

Since the typical multiplier, $1/[x(x+1)]^{1/2}$, is the reciprocal of the geometric mean between the successive integers x and $x+1$, the Method of Equal Proportions might be called also the *Method of the Geometric Mean.**

The proof of the correctness of the rule for Method EP is as follows.

Suppose that, in an apportionment made according to the rule, any State A has received $x+1$ representatives and any other State B has received y representatives; and suppose (as we may, without loss of generality) that

* The first use of the geometric mean in connection with this problem occurs in the *Method of Alternate Ratios*, proposed by Dr. J. A. Hill in 1910; this method, though obtained through entirely different reasoning (Tests 1 and 2 being unknown at that time), differs from the Method of Equal Proportions only in the fact that it insists on too close a relationship between the assignment given to any state and the true quota of that state. (This defect leads to an "Alabama paradox," as we shall see below.) Dr. Hill was also the first writer to recognize the superiority of the relative difference over the absolute difference, in the solution of this problem. See his paper in House of Representatives Report No. 12, of the Sixty-Second Congress, First Session, April 25, 1911.

State A is over-represented in comparison with State B. We proceed to show that if one representative is transferred from State A to State B, the "inequality" between the two states (measured according to Test 1 or Test 2) will be thereby increased.

We begin by showing that in the hypothetical apportionment, in which State A has x representatives and State B has $y+1$, the latter state will be over-represented in comparison with the former.

From the way in which the "priority list" is constructed we know that $A^2/[x(x+1)]>B^2/[y(y+1)]$; and since in the actual assignment A is over-represented in comparison with B, we know that $B/y>A/(x+1)$, and hence $(x+1)/A>y/B$. It follows that $B/(y+1)<A/x$, and hence $x/A<(y+1)/B$, since the contrary assumption would lead to contradiction. But these last relations express the fact that after the transfer is made, State B is over-represented in comparison with State A.

We can now write down the expression for the "inequality" between the two states before and after the transfer, remembering the convention that the "percentage difference" between any two numbers is understood to mean the "absolute difference divided by the smaller number."

In the actual assignment (before the transfer), the inequality in question is, by Test 1, $[B/y-A/(x+1)]/[A/(x+1)]$, or, by Test 2, $[(x+1)/A-y/B]/[y/B]$, each of which reduces to $B(x+1)/(Ay)-1$.

In the hypothetical assignment (after the transfer), the inequality is, by Test 1, $[A/x-B/(y+1)]/[B/(y+1)]$, or, by Test 2, $[(y+1)/B-x/A]/[x/A]$, each of which reduces to $A(y+1)/(Bx)-1$.

But from the given relation $A^2[x(x+1)]>B^2/[y(y+1)]$, we have at once $A^2(y+1)/x>B^2(x+1)/y$, whence

$$A(y+1)/(Bx)-1>B(x+1)/(Ay)-1,$$

which shows that the inequality between the two states would be increased by the transfer.

In other words, an apportionment made according to the rule, is one which cannot be "improved" (in the sense of Test 1 or Test 2), by any transfer of a representative from any state to any other state.

Critique of two conflicting methods

As pointed out above, whichever definition of the amount of inequality between two states may be adopted, it is clearly the relative or percentage difference, rather than the mere absolute difference, which is significant. The inappropriateness of the absolute difference is made still more apparent by the fact that its use leads us to two conflicting methods of apportionment.

Thus, if we substitute the "absolute difference" for the "relative difference" in Tests 1 and 2, we have the following tests:

TEST 1a (*not recommended*). *If the absolute difference between the two congressional districts, A/a and B/b, can be reduced by a transfer of a representative from one state to the other, then this transfer should be made.*

This test leads to a distinct method of apportionment, known as the *Method of the Harmonic Mean* (HM).*

TEST 2a (*not recommended*). *If the absolute difference between the two "individual shares," a/A and b/B, can be reduced by a transfer of a representative from one state to the other, then this transfer should be made (except that no state shall be left without at least one representative).*

This test leads to another distinct method of apportionment, known as the *Method of Major Fractions* (MF).†

Thus, while each of these tests is a "workable" test, each leads to a distinct method of apportionment. In comparison with the Method of Equal Proportions, the Method of the Harmonic Mean favors the small states unduly, while the Method of Major Fractions favors the large states unduly. This is illustrated in Example 1, showing the apportionment of 16 representatives among three states with a total population of 1600.

In this example, it is fairly obvious that State A should have at least 7 representatives, State B at least 5, and State C at least 3. But this makes only 15 in all. Where shall the 16th representative be assigned? Method HM gives it to the smallest state (C), and Method MF gives it to the largest state (A); while the Method of Equal Proportions gives it to the middle-sized state (B).

The computations in the right-hand part of the table will be self-explanatory. Method HM differs from Method EP only in regard to States B and C; the "amount of inequality" between these two states is smaller in

* See E. V. Huntington, in the paper already cited; or a brief abstract entitled *The mathematical theory of the apportionment of representatives*, in the Proceedings of the National Academy of Sciences, vol. 7, pp. 123–127, April, 1921.

† The Method of Major Fractions was devised by Professor W. F. Willcox in 1910, and was used in the apportionment for that year. See his paper in House of Representatives Report No. 12, of the Sixty-Second Congress, First Session, April 25, 1911, and his presidential address as president of the American Economic Association, published in the American Economic Review, vol. 6, no. 1, Supplement, pp. 1–16, March, 1916; also F. W. Owens, *On the apportionment of representatives*, Quarterly Publication of the American Statistical Association, December, 1921, pp. 958–968. The Report of the Advisory Committee cited above, concluded, after elaborate hearings, that the Method of Major Fractions was less desirable than the Method of Equal Proportions.

Example 1					Tests 1a and 1		Tests 2 and 2a	
State	Pop.	Assignment of Reps.			Size of Congr. Dist.		Size of Indiv. Share	
		HM	EP	MF	HM	EP	EP	MF
A	729	7	7	8			0.00960	0.01097
B	534	5	6	5	106.80	89.00	0.01124	0.00936
C	337	4	3	3	84.25	112.33		
	1600	16	16	16				
Absolute Difference					22.55	23.33	0.00164	0.00161
Relative Difference					0.268	0.262	0.170	0.172

column HM when Test 1a is used, and smaller in column EP when Test 1 is used. Similarly, Method MF differs from Method EP only in regard to States A and B; the "amount of inequality" between these two states is smaller in column MF when Test 2a is used, and smaller in column EP when Test 2 is used.

Thus Tests 1a and 2a lead to conflicting results (Methods HM and MF); if these were the only tests available, it would be difficult to make a choice between them on any but arbitrary grounds.

On the other hand, Tests 1 and 2 lead to no such dilemma, since the Method of Equal Proportions satisfies them both. This fact strengthens our belief that in defining a measure of inequality between two states, the relative difference is more natural and useful than the absolute difference.[*] The two conflicting Methods, HM and MF, may be regarded as on the same level of merit, as between themselves; but both of them are inferior to the Method of Equal Proportions.

Working rule for Method of Harmonic Mean

The working rule for the *Method of the Harmonic Mean* is the same as the rule for the Method of Equal Proportions, if, in forming the "priority list," we replace the series of multipliers, there given, by the following:

$$\frac{1}{2(1\cdot 2)/(1+2)},\quad \frac{1}{2(2\cdot 3)/(2+3)},\quad \frac{1}{2(3\cdot 4)/(3+4)},\ \cdots.$$

[*] See, however, Test 33, below, which uses either the absolute or the relative difference at pleasure.

The name Method of the Harmonic Mean is suggested by the fact that the typical multiplier, $1/[2(x)(x+1)/(x+x+1)]$, is the reciprocal of the harmonic mean between the successive integers x and $x+1$.

The proof that this rule will result in an apportionment satisfying Test 1a is as follows: Suppose, as before, that State A has $x+1$ representatives, and State B has y representatives, and that State A is over-represented in comparison with State B. Then the inequality between these two states, measured according to Test 1a, is $B/y-A/(x+1)$. If, on the other hand, State A had only x representatives and State B had $y+1$, then the inequality between the two states would be $A/x-B/(y+1)$. From the way in which the priority list is constructed, we know that $A(x+x+1)/[2(x)(x+1)] > B(y+y+1)/[2(y)(y+1)]$, whence $A/x+A/(x+1)>B/y+B/(y+1)$, whence $A/x-B/(y+1)>B/y-A/(x+1)$.

Working rule for the Method of Major Fractions

The working rule for the *Method of Major Fractions* is the same as the rule for the Method of Equal Proportions, with the replacement of the series of multipliers, there given, by the following:

$$\frac{1}{1+\frac{1}{2}},\quad \frac{1}{2+\frac{1}{2}},\quad \frac{1}{3+\frac{1}{2}},\ \ldots.$$

Since the typical multiplier, $1/(x+\frac{1}{2})$, is the reciprocal of the arithmetic mean between the successive integers x and $x+1$, the method might be called the *Method of the Arithmetic Mean.* The name "Method of Major Fractions," which is now well established, is due to Professor W. F. Willcox, who, approaching the subject from an entirely different point of view, devised the working rule, as a practical method of computation, in 1910, at a time when none of the theoretical tests (1, 2, 1a, 2a) were known. The Method of Major Fractions satisfies none of these tests except Test 2a.

The proof that this rule will result in an apportionment satisfying Test 2a is as follows: Suppose, as before, that State A has $x+1$ representatives, and State B has y representatives, and that State A is over-represented in comparison with State B. Then the inequality between those two states, measured by Test 2a, is $(x+1)/A-y/B$. If, on the other hand, State A had x and State B had $y+1$, then the inequality between the two states would be $(y+1)/B-x/A$. Now, from the way in which the priority list is constructed, $A/(x+\frac{1}{2})$ is greater than $B/(y+\frac{1}{2})$; hence $(2y+1)/B>(2x+1)/A$, whence $(y+1)/B-x/A>(x+1)/A-y/B$, which was to be proved.

Remarks on the name "Method of Major Fractions" and the "exact quota" of a state

As a question of practical politics, the controversy at the present time is chiefly between the Method of Equal Proportions (EP) and the Method of Major Fractions (MF).

To avoid any possible misinterpretation of the name "Method of Major Fractions," the following remarks are here inserted.

In a theoretically perfect apportionment, the exact quota of any state A is $A(R/P)$, where A is the population of the state, R is the total number of representatives in the House, and P is the total population of the country. If the exact quotas of all the states came out as whole numbers, the problem of apportionment would be solved without further ado. But in practically all cases, the exact quota will not be a whole number, and the actual assignment must be greater or less than the quota.

Now it is a common misconception that in a good apportionment the actual assignment should not differ from the exact quota by more than one whole unit; for example, if the exact quota is 5.21 or 5.76, then it is often assumed that the actual assignment should not be less than 5 nor more than 6.

It is a further misconception that if the exact quota is, say, 5 and a fraction, then if the fraction is less than 1/2 it should be disregarded, but if it is greater than 1/2, it should add one to the assignment. For example, it is often assumed that if the quota is 5.21, the assignment should be 5; and if the quota is 5.76, the assignment should be 6.

As a matter of fact, however, neither of these principles is a workable test of a good apportionment, and the Method of Major Fractions, like every other known method of apportionment, will often violate both of them.

Thus, in Example 2, both the Method of Major Fractions and the Method EP assign only 90 representatives to State A, although the exact quota of that state is 92.15.

Again, in Example 3, both methods assign 90 representatives to State A, although the exact quota of that state is only 87.85.

Further, in Example 4, the true quota of State A is 9.87; but both methods give State A only 9 representatives, in spite of the fact that the fraction 0.87 is very much greater than 1/2, and is, in fact, the largest of the three fractions which occur in this example.

Again, in Example 5, the true quota of State A is 7.31; but both methods give this state 8 representatives, in spite of the fact that the fraction 0.31

is less than 1/2, and is, in fact, the smallest of the three fractions that occur in this example.

Although crucial examples of this sort are not easy to construct, the existence of these examples is sufficient to show that *the "Method of Major Fractions" does not imply that a "major fraction" in the quota of a state will always entitle that state to an additional representative, or that a "minor fraction" is always to be disregarded.*

As a matter of fact, the size of the quota of an individual state, taken by itself, does not determine the number of representatives to which that state is entitled. For instance, in Examples 5 and 5a, the quota of State B is the same in both cases (5.35); and yet (according to either Method MF or Method EP), the number of representatives assigned to this state is 5 in one case and 6 in the other. This variation in the assignment given to State B is due not to any change in State B itself, but to a slight shift of population between the other two states.

Example 2

State	Pop.	EP MF
A	9215	90
B	159	2
C	158	2
D	157	2
E	156	2
F	155	2
	10,000	100

Example 3

State	Pop.	EP MF
A	8785	90
B	126	1
C	125	1
D	124	1
E	123	1
F	122	1
G	121	1
H	120	1
I	119	1
J	118	1
K	117	1
	10,000	100

Example 4

State	Pop.	EP MF
A	987	9
B	157	2
C	156	2
	1300	13

Example 5

State	Pop.	EP MF
A	731	8
B	535	5
C	334	3
	1600	16

Example 5a

State	Pop.	EP MF
A	729	7
B	535	6
C	336	3
	1600	16

Again, in Examples 6 and 6a, the quota for State B is exactly 44 in each case; but the actual assignment (according to either Method MF or Method EP) is 43 in one case and 45 in the other.

Example 6

State	Pop.	Quota	EP MF
A	5117	51.17	51
B	4400	44.00	43
C	162	1.62	2
D	161	1.61	2
E	160	1.60	2
	10,000	100	100

Example 6a

State	Pop.	Quota	EP MF
A	5189	51.89	52
B	4400	44.00	45
C	138	1.38	1
D	137	1.37	1
E	136	1.36	1
	10,000	100	100

Example 7

State	Pop.	Quota	No. of Reps.
A	1536	15.36	15
B	1535	15.35	15
C	1534	15.34	15
D	1533	15.33	15
E	1532	15.32	15
F	1530	15.30	15
G	162	1.62	2
H	161	1.61	2
I	160	1.60	2
J	159	1.59	2
K	158	1.58	2
	10,000	100	100

Group	Pop.	Quota	No. of Reps.
ABCDEF	9200	92.00	90
GHIJK	800	8.00	10
	10,000	100	100

Furthermore Example 7 shows that "nearness to the quota" with respect to groups of states is incompatible with "nearness to the quota" with respect to single states.

In this example, the quota of the group of large states is 92, while the actual assignment to this group is only 90; and the quota for the group of small states is 8, while the actual assignment to this group is 10. A transfer of a representative from the small group to the large group (say from State K to State A) would bring *both* groups "nearer to the quota"; and yet no one would wish to make this transfer.

In short, "nearness to the quota" cannot be taken as a test of a good assignment, either for a single state or for a group of states.

Remarks on the Willcox sliding divisor

The origin of the name "Method of Major Fractions" is to be found in an ingenious device known as the "*sliding divisor*," and due, in its present form, to Professor Willcox.

After an apportionment has been computed by the working rule for the Method of Major Fractions, the sliding devisor may be used to facilitate the recording of the results. This device is supplementary to the actual computation and forms no essential part of it; it is interesting chiefly as explaining the origin of the name.

The device consists in the selection of any number W such that

$$\frac{A}{a-\frac{1}{2}} > W > \frac{B}{a+\frac{1}{2}}, \quad \frac{B}{b-\frac{1}{2}} > W > \frac{B}{b+\frac{1}{2}}, \quad \frac{C}{c-\frac{1}{2}} > W > \frac{C}{c+\frac{1}{2}}, \text{ etc.,}$$

where a, b, c, etc., are the assignments of representatives to the States A, B, C, etc., according to Method MF.

Such a number W, which may be called a *Willcox Divisor*, will always exist,* and will have the following property: If the population of each State is divided by W, there will be obtained a series of quotients such that, if one representative is assigned for each unit and for each major fraction (and also for each quotient which is itself less than one-half), the resulting apportionment will be precisely the same as the apportionment given by the working rule for the Method MF, and will therefore satisfy Test 2a.

By the use of this device, the assignment given to any State can be figured out at once from the population of that State, as soon as the value of the Willcox divisor has been announced.

It should be noticed, however, that *the Willcox divisor is not the true value of the average Congressional District, and the Willcox quotients are not the true quotas of the several states*; hence the occurrence of a major fraction in the "quotient" of a State gives that State no claim whatever to an additional representative, except the claim which is already implied by Test 2a. If a method could be found which would assign an additional representative for every major fraction in the *true quota*, it would be indeed a simple and attractive method; but as we have seen, no such method is possible.

The Willcox "sliding divisor" merely provides a convenient way of recording the result of the method based on Test 2a, and adds nothing (except the name) to the authority of that method. The simplest basis for a valid method is a direct comparison between competing States, as expressed in Test 1 or Test 2; and the only method which satisfies either of these simple and natural tests is the Method of Equal Proportions.

Note on the Alabama paradox

The curious situation known as the "Alabama Paradox" is a further illustration of the confusion resulting from the unwise use of the exact quota of a state in computing the apportionment.

* Except in the case of a "tie" between two states (see below).

This paradox first came to the attention of Congress in the tables prepared in 1881, which gave Alabama 8 members in a House of 299, and only 7 members in a House of 300, so that an increase in the total size of the House actually produced a decrease in the number of representatives of one of the states.

The method in use at that time was known as the *Vinton Method of 1850*. This method assumed that each state was entitled to at least as many representatives as was indicated by the largest whole number contained in the exact quota of that state (with the special provision that no state should have less than one representative). To fill up the required total, further representatives were then assigned, "for fractions," to as many states as necessary, the states being arranged, for this latter purpose, in a "priority list," according to the magnitude of the fractions themselves, so that the state with the largest fraction was the first to receive an additional representative.

The resulting paradox is illustrated in Example 8, where State C has 11 representatives in a House of 100 members, and only 10 representatives in a House of 101.

A similar defect occurs in the otherwise excellent *Method of Alternate Ratios* proposed by Dr. J. A. Hill in 1910. This method proceeds as in the Vinton Method, except that the "priority list" for fractions is arranged according to the magnitude of the quantity $A/[x(x+1)]^{1/2}$, where x is the number already assigned to State A, and $x+1$ is the next larger number. The possible paradox resulting from this method is shown in Example 8a, where States G and H each lose one representative when the size of the House is increased from 100 to 101.

No method can be regarded as satisfactory which is subject to the Alabama Paradox.

Example 8
Vinton Method (Paradox)

	Pop.	100		101	
		Quota	Rep.	Quota	Rep.
A	453	45.3	45	45.753	46
B	442	44.2	44	44.642	45
C	105	10.5	11	10.605	10
	1000	100.0	100	101.000	101

Example 8a
Method of Alternate Ratios (Paradox)

	Pop.	100		101	
		Quota	Rep.	Quota	Rep.
A	154550	30.91	30	31.2191	31
B	154500	30.90	30	31.2090	31
C	154450	30.89	30	31.1989	31
D	7400	1.48	2	1.4948	2
E	7350	1.47	2	1.4847	2
F	7300	1.46	2	1.4746	2
G	7250	1.45	2	1.4645	1
H	7200	1.44	2	1.4544	1
	500000	100.00	100	101.0000	101

NOTE ON THE CASE OF A TIE BETWEEN TWO STATES

In applying the rule for the Method of Equal Proportions, the case of a "tie" between two states can occur only extremely rarely, that is to say, only when two "multipliers" (used in forming the priority list) happen to be commensurable numbers. For serial numbers up to $k=100$, this occurs only four times, as follows:

k	1/mult.	k	1/mult.	k	1/mult.	k	1/mult.
25	$10\,(6)^{1/2}$	49	$28\,(3)^{1/2}$	50	$35\,(2)^{1/2}$	81	$36\,(5)^{1/2}$
3	$(6)^{1/2}$	4	$2\,(3)^{1/2}$	9	$6\,(2)^{1/2}$	5	$2\,(5)^{1/2}$

The corresponding "ties" for the Method EP are as follows:

State	Pop.	(I)	(II)	State	Pop.	(I)	(II)	State	Pop.	(I)	(II)	State	Pop.	(I)	(II)
A	$10n$	25	24	C	$14n$	49	48	E	$35n$	50	49	G	$18n$	81	80
B	$1n$	2	3	D	$1n$	3	4	F	$6n$	8	9	H	$1n$	4	5
	$11n$	27	27		$15n$	52	52		$41n$	58	58		$19n$	85	85
$n=3, 4, 5, \cdots$.				$n=4, 5, 6, \cdots$.				$n=2, 3, 4, \cdots$.				$n=5, 6, 7, \cdots$.			

In each of these four cases, assignment (I) is chosen rather than assignment (II) merely on account of the convention which provides that in case of a tie preference shall be given to the state having the larger population.

On the other hand,in applying the rule for the Method of Major Fractions, the case of a tie may occur much more frequently. Thus, if p, q, and n are any positive integers, the following assignments (I) and (II) will always be tied in the Method MF:

State	Pop.	(I)	(II)
J	$(2p+1)n$	$p+1$	p
K	$(2q+1)n$	q	$q+1$
	$2(p+q+1)n$	$p+q+1$	$p+q+1$

We may even have a triple tie, as follows:

State	Pop.	Method MF (I)	Method MF (II)	Method MF (III)
L	11000	6	5	5
M	7000	3	4	3
N	3000	1	1	2
	21000	10	10	10

While none of these tie cases is likely to occur in actual practice in Congress, the extreme rarity of the possibility of such a tie in the Method of Equal Proportions is a theoretical argument in favor of that method.

Appendix I

Critique of two further conflicting methods

A third form in which the exact equality between two states may be written is

$$(\text{rep. over}) = (\text{rep. under})\frac{(\text{Pop. over})}{(\text{Pop. under})},$$

where "Pop. over" and "rep. over" stand for the population and number of representatives of that one of the two states which is over-represented in comparison with the other, and "Pop. under" and "rep. under" stand for the population and number of representatives of the under-represented state.

The (relative or absolute) difference between the two sides of this equation may be taken as a third measure of inequality between the two states, and may be called the (relative or absolute) *"representation-surplus"* belonging to the two states. If we use the relative difference, we obtain a third test, which we may call Test 3 (not written out here in detail), which leads to the Method of Equal Proportions. If we use the absolute difference, we have the following less desirable test:

Test 3a (*not recommended*). *If the absolute "representation-surplus" belonging to any two states, that is, the value of*

$$(\textit{rep. over}) - (\textit{rep. under})\,[(\textit{Pop. over})/(\textit{Pop. under})],$$

can be reduced by a transfer of a representative from one state to the other, then this transfer should be made.

This Test 3a proves to be a "workable" test, and leads to a distinct method of apportionment which may be called the *Method of Smallest*

Divisors (SD). In comparison with the Method of Equal Proportions, Method SD favors the small states even more than does the Method of the Harmonic Mean (see Example 9).

A fourth form of the exact equation, namely,

$$(\text{rep. over})\,\frac{(\text{Pop. under})}{(\text{Pop. over})} = (\text{rep. under}),$$

suggests, in a similar way, a Test 4, based on relative differences and leading to the Method of Equal Proportions, and a less desirable Test 4a, based on absolute differences, as follows:

TEST 4a (*not recommended*). *If the absolute "representation-deficiency" belonging to any two states, that is, the value of*

$$(\textit{rep. over})\,[(\textit{Pop. under})/(\textit{Pop. over})] - (\textit{rep. under}),$$

*can be reduced by a transfer of a representative from one state to the other, then this transfer should be made.**

This Test 4a proves to be "workable," and leads to another distinct method of apportionment which may be called the *Method of Greatest Divisors* (GD). In comparison with the Method of Equal Proportions, Method GD favors the large states even more than does the Method of Major Fractions (see Example 9).

Example 9					Tests 3a and 3		Tests 4 and 4a	
		Assignment of Reps.			Representation-Surplus		Representation-Deficiency	
			HM MF		$4-5C/B$	$6-3B/C$	$6A/B-7$	$8B/A-5$
State	Pop.	SD	EP	GD	SD	EP	EP	GD
A	726	7	7	8			7.000	5.939
B	539	5	6	5	3.108	6.000	8.082	5.000
C	335	4	3	3	4.000	4.827		
	1600	16	16	16				
Absolute					0.892	1.173	1.082	0.939
Relative					0.287	0.243	0.155	0.188

The conflict between Tests 3a and 4a, which does not exist between Tests 3 and 4, again confirms our belief that the relative difference is, for the present

* Tests 3a and 4a were presented by the present writer at a meeting of the American Mathematical Society on February 25, 1922.

problem, a more natural and useful idea than the absolute difference. The two conflicting Methods SD and GD may be regarded as on the same level of merit, as between themselves; but both of them are inferior to Methods HM and MF, and even more inferior to the Method of Equal Proportions.

Working rule for the Method of Smallest Divisors

The working rule for Method SD is the same as the rule for Method EP, except that, in forming the "priority list," the multipliers there given are replaced by those in the adjoining table.

Method SD	
No.	Multipliers
2	1/1
3	1/2
4	1/3
-	—
-	—

The proof that this rule satisfies Test 3a is as follows. Suppose, as before, that A has $x+1$ and B has y, and that State A is over-represented in comparison with State B. Then the inequality between States A and B, measured according to Test 3a, is $(x+1)-y(A/B)$. If, hypothetically, A had x and B had $y+1$, then the inequality would be $(y+1)-x(B/A)$. Now from the construction of the "priority list," $A/x>B/y$; hence

$$1 + (A + B)y/B > 1 + (A + B)x/A ;$$

hence

$$y + 1 + Ay/B > x + 1 + Bx/A,$$

whence

$$(y + 1) - x(B/A) > (x + 1) - y(A/B).$$

Working rule for the Method of Greatest Divisors

The working rule for Method GD is the same as the rule for Method EP, except that the table of "multipliers" is replaced by the table here given.

Method GD	
No.	Multipliers
2	1/2
3	1/3
4	1/4
-	—
-	—

The proof that this rule satisfies Test 4a is, briefly, as follows: If A has $x+1$ and B has y, the inequality, according to Test 4a, is $(x+1)B/A-y$. If A had x and B had $y+1$, the inequality would be $(y+1)A/B-x$. Now from the construction of the "priority list," $A/(x+1)>B/(y+1)$; hence

$$(A + B)(y + 1)/B > (A + B)(x + 1)/A,$$

whence

$$(y + 1)A/B + y + 1 > (x + 1)B/A + x + 1,$$

whence

$$(y + 1)A/B - x > (x + 1)B/A - y.^{*}$$

Comparison of the five known methods of apportionment

The only known methods of apportionment which are "workable," and avoid the Alabama Paradox, are the five methods described above, namely (in the order in which they favor the smaller states), Methods SD, HM; EP; MF, GD.

Example 10 (which is a combination of Examples 1 and 9) gives a comparison of the results of all five of the methods. Examples 11 and 12 are further examples which likewise separate the five methods. Of course in many cases, two or more of the methods will agree in their results.

Example 10

	Pop.	Number of Reps.				
		SD	HM	EP	MF	GD
A	729	7	7	7	8	8
B	726	7	7	7	7	8
C	539	5	6	6	6	5
D	534	5	5	6	5	5
E	337	4	4	3	3	3
F	335	4	3	3	3	3
	3200	32	32	32	32	32

Example 11

	Pop.	Number of Reps.				
		SD	HM	EP	MF	GD
A	762	7	8	8	8	8
B	758	7	7	7	8	8
C	555	6	5	6	5	6
D	351	4	4	3	3	3
E	174	2	2	2	2	1
	2600	26	26	26	26	26

Example 12

	Pop.	Number of Reps.				
		SD	HM	EP	MF	GD
A	9061	9	9	9	9	10
B	7179	7	7	7	8	7
C	5259	5	5	6	5	5
D	3319	3	4	3	3	3
E	1182	2	1	1	1	1
	26000	26	26	26	26	26

The following table gives a summary of the working rules for the five methods, arranged in the order in which they favor the small states.

It will be observed that the Method of Equal Proportions occupies the central position among the five methods, having no "bias" in favor of either the small or the large states.

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* The working rule for Method GD, except for the provision that every state shall have at least one representative, is the same as that devised by the Belgian, Victor d'Hondt, in 1885, as a practical method of computation; none of the theoretical tests (1, 2, 3, 4; 1a, 2a, 3a, 4a) were known at that time. On the history of the *d'Hondt Method*, see C. G. Hoag and G. H. Hallett, *Proportional Representation*, New York, 1926.

Between any two states, A and B, the assignment

A	$x+1$
B	y

is better than the assignment

A	x
B	$y+1$

provided

SD	$\dfrac{A}{x} > \dfrac{B}{y}$
HM	$\dfrac{A}{\dfrac{2x(x+1)}{x+x+1}} > \dfrac{B}{\dfrac{2y(y+1)}{y+y+1}}$
EP	$\dfrac{A}{[x(x+1)]^{1/2}} > \dfrac{B}{[y(y+1)]^{1/2}}$
MF	$\dfrac{A}{x+\frac{1}{2}} > \dfrac{B}{y+\frac{1}{2}}$
GD	$\dfrac{A}{x+1} > \dfrac{B}{y+1}$

Appendix II

Critique of certain unworkable tests

A fifth form in which the exact equation may be written is the following (using the notation explained above):

$$\frac{\text{rep. over}}{\text{rep. under}} = \frac{\text{Pop. over}}{\text{Pop. under}},$$

and the (relative or absolute) difference between these two numbers might be taken as the measure of inequality between the two states.

If we use the relative difference, the resulting Test 5 (which the reader may write out for himself) leads at once to the Method of Equal Proportions.

If, on the other hand, we use the absolute difference, we have a Test 5a which is not merely less desirable but is absolutely "unworkable."

Test 5a (*unworkable*). *If the value of the difference*

$$(\textit{rep. over})/(\textit{rep. under}) - (\textit{Pop. over})/(\textit{Pop. under})$$

belonging to any two states can be reduced by a transfer of representatives from one state to the other, then this transfer should be made.

In many cases this apparently plausible test fails to give any information as to which of several proposed apportionments is to be preferred. For example,

if we attempt to apply this test to the apportionment of 16 representatives to the three states whose populations are given in Example 13, we find that assignment (1) is better than assignment (2), and that (2) is better than (3), and also that (3) is better than (1), so that no choice is indicated.

Examples 14, 15, 16, 17, 18 establish in like manner the unworkableness of certain other tests, which will be listed below.

All these results confirm again our belief that the use of absolute differences, instead of the more natural relative differences, in this problem, is not well advised.

Ex. 13		Assignment of Reps.			(rep. over)/(rep. under) − (Pop. over)/(Pop. under)					
					$4/5-C/B$	$6/3-B/C$	$6/7-B/A$	$8/5-A/B$	$8/3-A/C$	$4/7-C/A$
	Pop.	(1)	(2)	(3)	(1)	(2)	(2)	(3)	(3)	(1)
A	762	7	7	8			0.857	1.600	2.667	0.571
B	534	5	6	5	0.800	2.000	0.701	1.427		
C	304	4	3	3	0.569	1.757			2.507	0.399
	1600	16	16	16	0.231	0.243	0.156	0.173	0.160	0.172

Ex. 14		Assignment of Reps.			(Pop. under)/(Pop. over) − (rep. under)/(rep. over)					
					$B/A-5/8$	$A/B-7/6$	$C/B-3/6$	$B/C-5/4$	$A/C-7/4$	$C/A-3/8$
	Pop.	(1)	(2)	(3)	(1)	(2)	(2)	(3)	(3)	(1)
A	698	8	7	7	0.7650	1.3071			1.897	0.527
B	534	5	6	5	0.6250	1.1667	0.689	1.451		
C	368	3	3	4			0.500	1.250	1.750	0.375
	1600	16	16	16	0.1400	0.1404	0.189	0.200	0.147	0.152

Ex. 15		Assignment of Reps			1/(rep. under) − [(Pop. over)/(Pop. under)]/(rep. over)					
					$1/5-(C/B)/4$	$1/3-(B/C)/6$	$1/7-(B/A)/6$	$1/5-(A/B)/8$	$1/3-(A/C)/8$	$1/7-(C/A)/4$
	Pop.	(1)	(2)	(3)	(1)	(2)	(2)	(3)	(3)	(1)
A	7496	7	7	8			.1429	.2000	.33333	.14286
B	5340	5	6	5	.20000	.33333	.1187	.1755		
C	3164	4	3	3	.14813	.28129			.29614	.10552
	16000	16	16	16	.05187	.05204	.0242	.0245	.03719	.03734

Ex. 16	Assignment of Reps.			[(Pop. under)/(Pop. over)]/(rep. under) − 1/(rep. over)					
				$(B/C)/5 - 1/4$	$(C/B)/3 - 1/6$	$(A/B)/7 - 1/6$	$(B/A)/5 - 1/8$	$(C/A)/3 - 1/8$	$(A/C)/7 - 1/4$
Pop.	(1)	(2)	(3)	(1)	(2)	(2)	(3)	(3)	(1)
A 7139	7	7	8			.1910	.1496	.1644	.2896
B 5339	5	6	5	.30318	.21989	.1667	.1250		
C 3522	4	3	3	.25000	.16667			.1250	.2500
16000	16	16	16	.05318	.05322	.0243	.0246	.0394	.0396

Ex. 17	Assignment of Reps.			|(rep. larger)/(rep. smaller) − (Pop. larger)/(Pop. smaller)|					
				$B/C-5/4$	$6/3-B/C$	$A/B-7/6$	$8/5-A/B$	$8/3-A/C$	$A/C-7/4$
Pop.	(1)	(2)	(3)	(1)	(2)	(2)	(3)	(3)	(1)
A 737	7	7	8			1.380	1.600	2.667	2.240
B 534	5	6	5	1.623	2.000	1.167	1.380		
C 329	4	3	3	1.250	1.623			2.240	1.750
1600	16	16	16	0.373	0.377	0.213	0.220	0.427	0.490

Ex. 18	Assignment of Reps.			|(rep. smaller)/(rep. larger) − (Pop. smaller)/(Pop. larger)|					
				$B/A-5/8$	$6/7-B/A$	$C/B-3/6$	$4/5-C/B$	$4/7-C/A$	$C/A-3/8$
Pop.	(1)	(2)	(3)	(1)	(2)	(2)	(3)	(3)	(1)
A 721	8	7	7	.7406	.8571			.571	.479
B 534	5	6	5	.6250	.7406	.646	.800		
C 345	3	3	4			.500	.646	.479	.375
1600	16	16	16	.1156	.1165	.146	.154	.092	.104

Summary of Tests 1–32 and Tests 1a–32a

A systematic examination of all the ways in which the exact equation may be written suggests a total of 64 measures of inequality between two states (32 based on the use of relative differences, and 32 based on the use of absolute differences; see table below).

The 32 tests based on relative differences may be called Tests 1–32, and

all lead to the same Method of Equal Proportions.* The 32 less desirable tests based on absolute differences may be called Tests 1a–32a, and lead to a confusion of miscellaneous results, as exhibited in the accompanying table.

UNDESIRABLE MEASURES OF INEQUALITY BETWEEN TWO STATES

In Tests 1a–16a, A = Pop. of over-represented state B = Pop. of under-represented state			In Tests 17a–32a, A = Pop. of larger state B = Pop. of smaller state		
(1a)	$\frac{B}{b}-\frac{A}{a}$	HM	(17a)	$\left\lvert\frac{B}{b}-\frac{A}{a}\right\rvert$	HM
(2a)	$\frac{a}{A}-\frac{b}{B}$	MF	(18a)	$\left\lvert\frac{a}{A}-\frac{b}{B}\right\rvert$	MF
(3a)	$a-\frac{A}{B}b$	SD	(19a)	$\left\lvert a-\frac{A}{B}b\right\rvert$	MF
(4a)	$\frac{B}{A}a-b$	GD	(20a)	$\left\lvert\frac{B}{A}a-b\right\rvert$	MF
(5a)	$\frac{a}{b}-\frac{A}{B}$	Ex. 13	(21a)	$\left\lvert\frac{a}{b}-\frac{A}{B}\right\rvert$	Ex. 17
(6a)	$B\frac{a}{b}-A$	SD	(22a)	$\left\lvert B\frac{a}{b}-A\right\rvert$	Ex. 17
(7a)	$\frac{1}{A}\frac{a}{b}-\frac{1}{B}$	SD	(23a)	$\left\lvert\frac{1}{A}\frac{a}{b}-\frac{1}{B}\right\rvert$	Ex. 17
(8a)	$\frac{B}{A}\frac{a}{b}-1$	EP	(24a)	$\left\lvert\frac{B}{A}\frac{a}{b}-1\right\rvert$	Ex. 17
(9a)	$1-\frac{A}{B}\frac{b}{a}$	EP	(25a)	$\left\lvert 1-\frac{A}{B}\frac{b}{a}\right\rvert$	Ex. 18
(10a)	$B-A\frac{b}{a}$	GD	(26a)	$\left\lvert B-A\frac{b}{a}\right\rvert$	Ex. 18
(11a)	$\frac{1}{A}-\frac{1}{B}\frac{b}{a}$	GD	(27a)	$\left\lvert\frac{1}{A}-\frac{1}{B}\frac{b}{a}\right\rvert$	Ex. 18
(12a)	$\frac{B}{A}-\frac{b}{a}$	Ex. 14	(28a)	$\left\lvert\frac{B}{A}-\frac{b}{a}\right\rvert$	Ex. 18
(13a)	$\frac{1}{b}-\frac{A}{B}\frac{1}{a}$	Ex. 15	(29a)	$\left\lvert\frac{1}{b}-\frac{A}{B}\frac{1}{a}\right\rvert$	HM
(14a)	$\frac{B}{A}\frac{1}{b}-\frac{1}{a}$	Ex. 16	(30a)	$\left\lvert\frac{B}{A}\frac{1}{b}-\frac{1}{a}\right\rvert$	HM
(15a)	$Ba-Ab$	MF	(31a)	$\lvert Ba-Ab\rvert$	MF
(16a)	$\frac{1}{Ab}-\frac{1}{Ba}$	HM	(32a)	$\left\lvert\frac{1}{Ab}-\frac{1}{Ba}\right\rvert$	HM

Note. (17a) = (1a), (18a) = (2a), (31a) = (15a), (32a) = (16a).

* Tests 1–32 may be read immediately from the Table of Tests 1a–32a by dividing each difference by the smaller of its two terms. The resulting relative difference will be equal to $(a/b)/(A/B)-1$ or $1-(A/B)/(a/b)$, according as A or B is the over-represented state.

In this table, in Tests 1a–16a, A stands for the over-represented state, and B for the under-represented state; and in Tests 17a–32a, A stands for the larger state, and B for the smaller, while the vertical bars indicate that the absolute value of the quantity is to be taken, without regard to sign. Tests followed by "Ex. 13," "Ex. 14," etc., are "unworkable" tests, the proof of this fact being supplied in each case by the example cited.

It may be noted that measures 17a and 18a are the same as measures 1a and 2a, respectively, while measures 31a and 32a are the same as 15a and 16a. Measures 17a, 29a, 30a, 32a differ only by a constant factor; and the same is true of measures 18a, 19a, 20a, 31a, and of measures 21a, 22a, 23a, 24a, and of measures 25a, 26a, 27a, 28a.

A further basis for the method of equal proportions

It may also be noted that Tests 8a and 9a, although based on absolute differences, happen to lead to the Method of Equal Proportions. The quantity $(a/b)/(A/B)-1$, which occurs in Test 8a, and the quantity $1-(A/B)/(a/b)$, which occurs in Test 9a, may be called, respectively, the (absolute) *ratio-surplus* and the (absolute) *ratio-deficiency* belonging to the two states. If we use the term *ratio-discrepancy* to mean, at pleasure, the relative or absolute ratio-surplus, or the relative or absolute ratio-deficiency, belonging to two states, then the four tests 8, 9, 8a, 9a may be combined into a single criterion, as follows:

Test 33. *If the "ratio-discrepancy" belonging to any two states (that is, the relative or absolute amount by which $(a/b)/(A/B)$ or $(A/B)/(a/b)$ differs from unity) can be reduced by a transfer of a representative from one state to the other, then this transfer should be made.*

This Test 33, like all the Tests 1–32, leads directly to the Method of Equal Proportions. The original Tests 1 and 2 remain, however, perhaps the most satisfactory characterization of the Method.

Briefly, the Method of Equal Proportions may be described as the only method which makes (1) *the ratio of population to representatives and* (2) *the ratio of representatives to population, as nearly uniform as possible among the the several states.*

Appendix III

Critique of methods based on average or total error

All the discussion up to this point has been based on the idea of comparison between competing states, and all the tests so far considered may be called

"comparison tests." There is another possible method of approach to the problem, however, which should here be mentioned. This is based on the idea of computing some sort of average or total error for the whole apportionment, and selecting as the best apportionment that one whose *total error* is the least.

There are two objections to this method of approach. In the first place, it is obvious that a total or average error might be reasonably small, while at the same time the error affecting some particular state might be shockingly large; and a gross injustice done to a particular state could hardly be successfully defended on the ground that "on the average" the other states are fairly treated.

In the second place, when one actually tries to set up a definition for the total or average error, the multiplicity of possible formulas makes it extremely difficult to select any one as more significant than the rest. If q is the true quota, and r the actual number of representatives, of the ith state, then the error attached to that particular state may be defined in at least four different ways: $r-q$, $r/q-1$, $1-q/r$, $1/q-1/r$; and the total error may then be defined as the simple sum or as the weighted sum, of either the absolute values of the errors, or the squares of the errors; and the weighting factors may be chosen in a great variety of ways. Most of the resulting methods can be shown to involve the Alabama Paradox; the only ones which do not, lead about equally to the Method of Major Fractions and the Method of Equal Proportions.

Thus, Method MF minimizes

$$\Sigma\left[q\left(\frac{r}{q}-1\right)^2\right]=\Sigma\left[r\left(\frac{r-q}{(rq)^{1/2}}\right)^2\right]=\Sigma\left[\frac{(r-q)^2}{q}\right],$$

while Method EP minimizes

$$\Sigma\left[r\left(1-\frac{q}{r}\right)^2\right]=\Sigma\left[q\left(\frac{r-q}{(rq)^{1/2}}\right)^2\right]=\Sigma\left[\frac{(r-q)^2}{r}\right].$$

As neither of these sets of formulas appears to have any obvious advantages over the other, it is difficult to make out a clear case for either the Method MF or the Method EP on the basis of the idea of total error.*

Finally, it may occur to one to use, as the measure of error of the whole apportionment, not the sum of the errors of the several states, but the maximum error with which any state is affected; the best apportionment being

* See F. W. Owens, loc. cit., and E. V. Huntington, loc. cit.

that one which has the smallest ***maximum error***. As far as is known, all attempts to apply this principle lead to the Alabama Paradox.

We are thrown back, therefore, on the simple comparison tests, the study of which reveals the substantial advantages of the Method of Equal Proportions.

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CONDITIONS FOR ASSOCIATIVITY OF DIVISION ALGEBRAS CONNECTED WITH NON-ABELIAN GROUPS*

BY

JOHN WILLIAMSON

1. **Introduction.** The problem of the determination of division algebras has been successfully investigated by Professor L. E. Dickson, who was the first to discover a division algebra D of order n^2 over a field F, and who recently has shown† how to construct all algebras Γ of order $n^2 = Q^2q^2$ over a field F, corresponding to the Galois group G of order $n = Qq$ of an equation $f(x) = 0$ irreducible in F. In addition, he has determined the general conditions, called D_1, D_2, D_3, which must be satisfied by the algebra Γ if it is associative. He has also reduced these conditions in detail for an algebra Γ corresponding to a Galois group G of two generators, both when G is an abelian group, and when G is not abelian but of a special type. Based on his work, our problem is to reduce the associativity conditions, first when the group G is generated by two generators Θ_q and Θ_1, where Θ_q transforms Θ_1 into some power of Θ_1; and second when G is generated by three generators Θ_q, Θ_p, and Θ_1, where Θ_p transforms Θ_1 into some power of Θ_1 and Θ_q transforms Θ_1 and Θ_p into powers of Θ_1 and Θ_p respectively.

As this paper is a continuation of Dickson's paper (these Transactions, vol. 28 (1926), pp. 207–234), direct reference is made to it throughout. Numbered lemmas, numbered theorems and formulas in square brackets refer to lemmas, theorems and formulas in his paper. The notation is everywhere the same except that, for convenience in this paper, Q has been used for p, and δ for β. It is assumed that the reader has Dickson's paper before him.

The conditions D_1, D_2, and D_3 are the formulas [53], [55], and [58] of Theorem 10:

$$D_1 \qquad \delta = \delta(\theta_q)\alpha_e,$$

$$D_2 \qquad \alpha_k\alpha_r(\theta_{k_0})c_{k_0r_0} = c_{kr}(\theta_q)\alpha_u \qquad (k, r = 1, \cdots, q-1;\ \alpha_0 = 1),$$

$$D_3 \qquad \delta d_k = \alpha_k(\theta_q^{Q-1})\alpha_{k_0}(\theta_q^{Q-2})\alpha_{k_{00}}(\theta_q^{Q-3}) \cdots \alpha_{k_0\cdots{}_0}\delta(\theta_{k_0\cdots{}_0})$$
$$(k = 1, 2, \cdots, q-1),$$

* Presented to the Society, December 31, 1926; received by the editors January 28, 1927.

† L. E. Dickson, *New division algebras*, these Transactions, vol. 28 (1926).

where there are $Q-1$ subscripts 0 under the last α and Q under the final θ.

PART 1. ALGEBRAS Γ CONNECTED WITH A NON-ABELIAN GROUP GENERATED BY TWO GENERATORS

2. **The group G.** Let G_q be the cyclic group generated by Θ_1 of order q, and let G_q be extended to G by Θ_q where G_q is of index Q under G. Then the Qth, but no lower than the Qth power of Θ_q, is a substitution of G_q. If also Θ_q transforms Θ_1 into some power x of Θ_1, then

$$\Theta_q^Q = \Theta_1, \quad \Theta_q^{-1}\Theta_1\Theta_q = \Theta_1^x$$

where e and x are integers less than q.

Since G_q is cyclic we may denote $\Theta_1{}^k$ by $\Theta_k(k<q)$ and hence

$$\Theta_q^{-s}\Theta_k\Theta_q{}^s = \Theta_1{}^{kx^s} \text{ for all integers } s > 0. \tag{1}$$

But $\Theta_e = \Theta_q^Q$ and is commutative with Θ_q, hence it follows from (1) with $k=e$ and $s=1$ that

$$e(x-1) \equiv 0 \pmod{q}. \tag{2}$$

For the same reason replacing s by Q and k by 1 in (1), we see that

$$x^Q \equiv 1 \pmod{q} \tag{3}$$

and that x is relatively prime to q. Groups of this type exist; one such is a transitive group of order 16 with $Q=2$, $q=8$, $x=5$ and $e=4$.

3. **Algebra Σ.** The units j may be given the notation

$$j_1^k = j_k, \quad j_q^s = j_{sq}, \qquad j_kj_{sq} = j_{sq+k} \qquad (k<q, \quad s<Q), \tag{4}$$

$$j_1{}^q = g, \quad j_q^Q = \delta_{i_e}, \tag{5}$$

where g and δ are numbers $\neq 0$ of $F(i)$. We also see that

$$k_0 \equiv kx \pmod{q}, \quad k_{0\cdots 0} \equiv kx^s \pmod{q} \qquad (k_0 < q, \quad k_{0\cdots 0} < q)$$

where there are s zeros as subscript to k. Throughout this part of the paper $\alpha_{k_{s0}}$ will denote $\alpha_{k_{0,\dots,0}}$ where there are s zeros subscript to k.

The subgroup G_q is now cyclic. Hence by Theorem 1 the algebra Σ may be regarded as an algebra of order q^2 over the field F_1, derived from F by adjoining all the symmetric functions of i, $\theta_1(i), \cdots, \theta_{q-1}(i)$. This algebra is associative if $g=g(\theta_1)$.* Consequently, by Theorem 10, Γ is associative, if the conditions D_1, D_2 and D_3 all hold and $g=g(\theta_1)$.

* Loc. cit., §4.

4. **Associativity conditions for Γ.** Equation [6] gives the following formulas:

$$
(6)\qquad
\begin{aligned}
j_k j_r &= j_u, & u &= k + r, & c_{kr} &= 1 & &(r + k < q),\\
j_{kr} &= g j_u, & u &= k + r - q, & c_{kr} &= g & &(r + k \geqq q,\ r < q,\ k < q).
\end{aligned}
$$

The condition D_1 gives

$$
(7)\qquad \delta = \delta(\theta_q)\alpha_e.
$$

Let us now consider the condition D_2. For any integer $m>0$, there exists an integer a_m, $0\leqq a_m<q$, and an integer $t_m>0$ such that $t_m x=mq+a_m$ and $(t_m-1)x<mq$. We define t_0 to be 1. Hence a_m is the value of t_{m_0}, which is written for $(t_m)_0$. If $t_{m+1}>k\geqq t_m$, then $k=t_m+s$, $kx=mq+a_m+sx$ and $k_0=a_m+sx$. In the same way, if $t_{n+1}>r\geqq t_n$, $r=t_n+v$ and $r_0=a_n+vx$.

If $k+r<q$, $c_{kr}=1$ by (6) and, if $k_0+r_0<q$, $c_{k_0r_0}=1$ and $(k+r)x=(m+n)q+r_0+k_0$. Consequently $u=t_{m+n}+b$ and D_2 becomes

$$
(8)\qquad \alpha_{t_m+s}\alpha_{t_n+v}(\theta_1^{kx}) = \alpha_{t_{m+n}+b}.
$$

But, if $k_0+r_0\geqq q$, $u=t_{m+n+1}+b$, and D_2 becomes

$$
(9)\qquad \alpha_{t_m+s}\alpha_{t_n+v}(\theta_1^{kx})g = \alpha_{t_{m+n+1}+b}.
$$

If we write $k=1$, that is $m=0$ and $s=1$, in (8) and (9) we get

$$
(10)\qquad \alpha\,\alpha_{t_n+v}(\theta_1^{x}) = \alpha_{t_n+v+1} \qquad (v+1<t_{n+1}-t_n),
$$

$$
(11)\qquad \alpha\,\alpha_{t_n+v}(\theta_1^{x})g = \alpha_{t_{n+1}} \qquad (v+1=t_{n+1}-t_n),
$$

and (12) follows by induction from (10) and (11):

$$
(12)\qquad \alpha_r = \alpha_{t_n+v} = g^n\alpha\alpha(\theta_1^{x})\cdots\alpha(\theta_1^{(r-1)x}) \qquad (r=1,2,\cdots,q-1).
$$

It is easily verified that equations (9) and (10) are satisfied identically, when the values for α_k, α_r and α_u from (12) are substituted into them.

When $k+r=q$, $k_0+r_0=q$ and so $c_{kr}=c_{k_0r_0}=g$, while $u=0$. Hence D_2 becomes $\alpha_k\alpha_r(\theta_1^{kx})g=g(\theta_q)$, or on substitution for α_k and α_r from (12)

$$
(13)\qquad \alpha\alpha(\theta_1^{x})\alpha(\theta_1^{2x})\cdots\alpha(\theta_1^{(q-1)x})g^{x} = g(\theta_q).
$$

That g occurs on the left hand side to the power of x is easily seen. For

$$
(k+r)x = (m+n)q + k_0 + r_0 = (m+n+1)q,
$$

$$
m+n+1 = x.
$$

If $k+r>q$, $c_{kr}=g$ and $u=k+r-q$. Then, as in the previous cases,

$$\begin{aligned} c_{k_0r_0} &= 1, \quad k+r = t_{m+n}+b, \quad u = t_{m+n-z}+b\,; \\ &= g, \quad k+r = t_{m+n+1}+b, \quad u = t_{m+n+1-z}+b. \end{aligned}$$

On substituting for α_k, α_r and α_u their values from (12) into D_2 and cancelling the terms common to both sides, we see that, when $k+r>q$, D_2 reduces to (13). Hence we have the following lemma:

LEMMA A. *The condition D_2 reduces for all values of $k, r<q$, to* (12) *or* (13), *where* (12) *merely serves to express $\alpha_r(r=2, \cdots, q-1)$ in terms of α.*

Next, let us consider the condition D_3. Since $X^Q \equiv 1 \pmod q$, $j_{k_0 \ldots 0}=j_k$ (where there are Q subscripts 0) and, since j_e and j_k are commutative, d_k in D_3 is equal to 1. Condition D_3 becomes

$$(14) \qquad \delta = \alpha_k(\theta_q^{Q-1})\alpha_{kz}(\theta_q^{Q-2}) \cdots \alpha_{kz^{Q-1}}\delta(\theta_1^k) \qquad (k = 1, 2, \cdots, q-1).$$

LEMMA B. *The condition* (14) *follows for all values of $k<q$ from*

$$(15) \qquad \delta = \alpha(\theta_q^{Q-1})\alpha_z(\theta_q^{Q-2})\alpha_{z^2}(\theta_q{}^{Q-3}) \cdots \alpha_{z^{Q-1}}\delta(\theta_1).$$

To prove this lemma by induction, we assume that (14) holds for all values of $k \leqq k$ and, writing $\theta_1{}^k$ for i in (15), combine the equation thus obtained with (14). Since by [8] and (1)

$$\theta_q^{Q-s}\theta_1^k = \theta_1^{kz^s}\theta_q^{Q-s},$$

$$\delta\, \delta(\theta_1{}^k) = \prod_{s=1}^{s=Q} \alpha_{kz^{s-1}}(\theta_q^{Q-s})\alpha_{z^{s-1}}(\theta_1{}^{kz^s}\theta_q^{Q-s})\delta(\theta_1{}^k)\delta(\theta_1{}^{k+1}).$$

But by the general formula D_2 this becomes

$$(16) \qquad \delta = \delta(\theta_1{}^{k+1}) \prod_{s=1}^{s=Q} \alpha_{(k+1)z^{s-1}}(\theta_q^{Q-s}) \frac{c_{kz^{s-1},z^{s-1}}(\theta_q^{Q-s+1})}{c_{kz^s,z^s}(\theta_q^{Q-s})},$$

(Since $\Theta_1^{kz^s}=\Theta_{k_0\ldots 0}$, c_{kz^s,z^s} is used to denote $c_{k_0\ldots 0,1_0\ldots 0}$, where there are s subscripts 0.). All the c's in this product cancel except the first of the numerator and the last of the denominator, namely $c_{k,1}(\theta_q^Q)$ and c_{kz^Q,z^Q}, each of which is equal to 1, since for the induction $k<q-1$. Hence (16) is simply (14) with k replaced by $k+1$. As (14) holds for $k=1$ the proof of the lemma is complete.

We have now proved the following theorem:

THEOREM A. *Let $f(x)=0$ be an equation of degree Qq irreducible in F whose Galois group G is generated by Θ_1 and Θ_q, such that Θ_1 is of order q and Θ_q transforms Θ_1 into Θ_1^z and $\Theta_q{}^Q=\Theta_1{}^e$, while no lower than the Qth power of*

Θ_q is equal to a power of Θ_1. Excluding the case $q=2$, we see that G is not abelian and that x, e, q and Q must satisfy (2) *and* (3). *The roots of $f(x)=0$ are*

$$\theta_1^k(\theta_q^r(i)) = \theta_q^r(\theta_1^{kx^r}(i)) \qquad \begin{pmatrix} r = 0,1,\cdots,Q-1 \\ k = 0,1,\cdots,q-1 \end{pmatrix},$$

where $\theta_1^q(i)=i$, $\theta_q^Q(i)=\theta_1^e(i)$, and θ_1 and θ_q are rational functions of i with coefficients in F. There exists an associative algebra Σ whose elements are

$$A = f_0 + f_1 j_1 + f_2 j_1^2 + \cdots + f_{q-1} j_1^{q-1},$$

where the f_k are polynomials in i of degree less than Qq with coefficients in F, while

$$j_1^q = g(i) = g(\theta_1), \quad j_1^r\phi(i) = \phi(\theta_1^r(i))j_1^r \qquad (r = 1,\cdots,q-1),$$

so that the product of any two elements of Σ is another element of Σ. Let

$$A' = f_0(\theta_q) + \sum_{k=1}^{q-1} f_k(\theta_q)\alpha_k j_{xk},$$

where α_k is defined by (12). *Then under multiplication defined by* [20] *the totality of polynomials in j_q with coefficients in Σ form an algebra of order Q^2q^2 over F, which is associative if and only if $g=g(\theta_1)$, $\delta=\delta(\theta_q)\alpha_e$, and* (13) *and* (15) *hold.*

Part 2. Algebras Γ connected with a group generated by three generators

5. **The group G.** Let the group G have the invariant subgroup G_q, which is of the same type as the group G considered in §2, where G_q has the invariant cyclic subgroup G_p generated by Θ_1 of order p, and G_p is of index P under G_q and is extended to G_q by the substitution Θ_p. Further, let G_q be of index Q under G so that the Qth, but no lower than the Qth, power of Θ_q is a substitution of G_q. Then, if Θ_q transforms Θ_1 into Θ_1^y and Θ_p into Θ_p^z, while Θ_p transforms Θ_1 into Θ_1^x, we have

(17) $\quad \Theta_q^Q = \Theta_{e'} = \Theta_p^{e_2}\Theta_1^{e_1}, \quad \Theta_p^P = \Theta_e = \Theta_1^e \qquad (e < p,\ e_1 < p,\ e_2 < P),$

(18) $\quad \Theta_p^{-s}\Theta_1^a\Theta_p^s = \Theta_1^{ax^s}.$

(19) $\quad \Theta_q^{-s}\Theta_1^a\Theta_q^s = \Theta_1^{ay^s},$

(20) $\quad \Theta_q^{-s}\Theta_p^b\Theta_q^s = \Theta_p^{bz^s},$

where a, b and s are integers >0.

It follows from §2 that the substitutions of G_q are represented uniquely in the form $\Theta_k=\Theta_{bp+a}=\Theta_p{}^b\Theta_1{}^a$ $(b<P,\ a<p)$ and if $q=Pp$ the substitutions of G in the form $\Theta_{rq+k}=\Theta_q{}^r\Theta_k(r<Q,\ k<q)$. As in §2 we see that

$$(21)\qquad x^P \equiv 1 \qquad \pmod{p},$$

$$(22)\qquad (x-1)e \equiv 0 \qquad \pmod{p}.$$

If we write $s=Q$, $a=1$ in (19), it follows from (17) that

$$(23)\qquad x^{e_2} \equiv y^Q \qquad \pmod{p}.$$

Similarly, from (17) and (20) with $s=Q$, we find that

$$(24)\quad b(z^Q-1)=bmP,\ emb+e_1(x^b-1)\equiv 0 \quad \pmod{p}(b=1,\cdots,P-1).$$

But (24) is satisfied if

$$(25)\qquad z^Q-1=mP,\ em+e_1(x-1)\equiv 0 \qquad \pmod{p}(m \text{ integer} > 0).$$

In addition the transforms of Θ_q^Q and $\Theta_p^{e_2}\Theta_1^{e_1}$ by Θ_q must be equal and also the transforms of Θ_p^P and Θ_e by Θ_p. Hence we have

$$(26)\qquad e_2(z-1)=np,\quad e(z-y)\equiv 0 \qquad \pmod{p}(n \text{ integer} > 0).$$

Finally, since

$$\Theta_q^{-1}(\Theta_p^{-1}\Theta_1\Theta_p)\Theta_q = (\Theta_q^{-1}\Theta_p^{-1})\Theta_1(\Theta_p\Theta_q),$$
$$\Theta_1{}^{xy} = \Theta_1{}^{yx^z},$$

and, as x is relatively prime to p, y is relatively prime to p by (23). Hence

$$(27)\qquad x^{z-1} \equiv 1 \qquad \pmod{p}.$$

Other conditions to be satisfied by the parameters e, e_1, e_2, x, y, and z may be deduced, but these are all that will be required. It is sufficient for our purpose that groups of this type do exist. For example, there is a transitive group of order 32 in which $p=4$, $P=4$, $Q=2$, $e=2$, $e_1=2$, $e_2=0$ and $x=y=z=3$.

If $k=a+bp(a=0,1,\cdots,p-1;b=0,1,\cdots,P-1)$, then $k_{00\cdots0}=a_{00\cdots0}+b_{00\cdots0}p$ where $a_{00\cdots0}<p$ and $\equiv ay^s \pmod{p}$, $b_{00\cdots0}<P$ and $\equiv bz^s \pmod{P}$ and there are s subscripts 0. With these values of k and k_0, the units and constants of multiplication of Γ are given by formulas [49], [50] and [52], where p, e and β are replaced by Q, e' and δ respectively.

6. **The algebra Σ.** The subgroup G_q being now of the type G considered in §2, the algebra Σ, which by Theorem 1 may be regarded as an algebra of order q^2 over the field F_1, derived from F by adjoining all the symmetric functions of i, $\theta_1(i),\cdots,\theta_{q-1}(i)$, is of the type Γ considered in Part 1. If

we substitute p, P, β and ρ for q, Q, α and δ respectively, all the formulas of Part 1 hold. Hence Σ is associative if, and only if,

$$
\begin{aligned}
& g = g(\theta_1), \\
& \rho = \rho(\theta_p)\beta_e, \\
(28) \qquad & \beta\,\beta(\theta_1{}^{z})\beta(\theta_1^{2z}) \cdots \beta(\theta_1{}^{(p-1)z})g^z = g(\theta_p), \\
& \rho = \beta(\theta_p^{P-1})\beta_z(\theta_p^{P-2})\beta_{z^2}(\theta_p^{P-3}) \cdots \beta_{z^{p+1}}\rho(\theta_1).
\end{aligned}
$$

By Theorem 10, if (28) holds, Γ is associative if and only if the conditions D_1, D_2, and D_3 all hold. In these conditions, as quoted in the introduction, we must now write e' for e.

7. **Associativity conditions for Γ.** Condition D_1 gives

$$(29) \qquad \delta = \delta(\theta_q)\alpha_{e'} \qquad (e' = e_1 + e_2 p).$$

In the consideration of condition D_2, let

$$
\begin{aligned}
k &= bp + a \\
r &= sp + t
\end{aligned}
\qquad
\begin{pmatrix} a,\ t = 0,1,\cdots,p-1 \\ b,\ s = 0,1,\cdots,P-1 \end{pmatrix}.
$$

If $b=s=0$, we see as in §4 that D_2 reduces to (30) and (31):

$$(30) \qquad \alpha_a = \alpha_{t_n+v} = g^n\alpha\alpha(\theta_1^{\,y}) \cdots \alpha(\theta_1{}^{(a-1)y}) \qquad (a = 1,2,\cdots,p-1),$$

$$(31) \qquad g(\theta_q) = \alpha\alpha(\theta_1^{\,y}) \cdots \alpha(\theta_1{}^{(p-1)y})g^y,$$

where $yt_n = np + a_n$ and $(t_n - 1)y < np$, while $t_{n+1} > a \geqq t_n$.*

Now, let $a=t=0$ so that k and r are multiples of p and may be taken as kp and rp respectively. Hence we must consider the condition

$$(32) \qquad \alpha_{kp}\alpha_{rp}(\theta_{kp_0})c_{kp_0,rp_0} = c_{kp,rp}(\theta_q)\alpha_u.$$

If $zt_m = mP + a_m, z(t_m - 1) < mP (m = 0,1,\cdots,z-1)(a_m < P)$,* and $t_{m+1} > k \geqq t_m$, then $k = t_m + s$ and $kz = mP + b$, where $b = sz + a_m < P$.

Since, by the second of (17), $\Theta_p^{kz} = \Theta_p^b\Theta_1^{me}$,† we must consider the value of em. As at the beginning of §4 we can find integers f_μ and $a_\mu \geqq 0$, such that $ef_\mu = \mu p + a_\mu$ and $e(f_\mu - 1) < p$ where $a_\mu < p$. Then, if $f_{\mu+1} > m = f_\mu + h \geqq f_\mu$, $\Theta_p^{kz} = \Theta_p^b\Theta_1^{a_\mu + he}$. Hence $kp_0 = bp + a_\mu + he$. Similarly, if $r = t_n + v$, $n = f_\nu + w$,

* See the definition of t_m and a_m at the beginning of §4.

† If $e=0$ the work is exactly similar to that in §4.

then $rp_0=dp+a_\nu+we$, where $d=vz+a_n<P$. We now require to consider the value of c_{kp_0,rp_0}. Since

$$\begin{aligned} j_{kp_0}j_{rp_0} &= c_{kp_0,rp_0}j_{u_0} \\ &= j_1^{a_\mu+he}j_p^{b}j_1^{a_\nu+we}j_p^{d}, \end{aligned}$$

then

$$c_{kp_0,rp_0}j_{u_0} = c_{bp,ne}(\theta_1^{me})j_1^{\sigma}j_p^{d+b}, \tag{33}$$

where $\sigma=a_\mu+a_\nu+(h+w)e$.

For, since

$$a_\nu + we \equiv ne \pmod{p},$$

$$(a_\nu + we)x^b \equiv nex^b \pmod{p}$$

and so by (22)

$$nex^b \equiv ne \equiv a_\nu + we \pmod{p}.$$

In (33), $c_{bp,ne}$ denotes $c_{bp,f}$, where $ne\equiv f \pmod{p}$ and $f<p$, and later, to simplify the formulas, $c_{bp+a,sp+t}$ is often written for c_{kr}, if $\Theta_q^b\Theta_1^a=\Theta_k$ and $\Theta_q^s\Theta_1^t=\Theta_r$, even when a and t are greater than p, and b and s greater than P. When $b+d<P$, $j_p^{b+d}=j_{(b+d)p}$ and, if $\sigma<p$, $m+n$ is of the form $f_{\mu+\nu}+t$ and $c_{kp_0,rp_0}=c_{bp,ne}(\theta_1^{me})$; but, if $\sigma\geqq p$, then $m+n$ is of the form $f_{\mu+\nu+1}+t$ and $c_{kp_0,rp_0}=gc_{bp,ne}(\theta_1^{me})$.

When $b+d\geqq P$, $j_p^{b+d}=\rho j_1^{e}j_p^{b+d-P}$, and from (33) we see that a factor g or g^2 occurs in c_{kp_0,rp_0}, according as $\sigma+e\geqq p$ or $\geqq 2p$; that is, according as $m+n+1$ is of the form $f_{\mu+\nu+1}+t$ or $f_{\mu+\nu+2}+t$. Hence the complete values of c_{kp_0,rp_0} as obtained from (33) are given by

$$c_{kp_0,rp_0} = Xc_{bp,ne}(\theta_1^{me}) \tag{34}$$

where

$$\begin{aligned} X &= 1, \text{ if } k+r=t_{m+n}+s,\ m+n=f_{\mu+\nu}+t, \\ &= g, \text{ if } k+r=t_{m+n}+s,\ m+n=f_{\mu+\nu+1}+t, \\ &= \rho(\theta_1^{(m+n)e}), \text{ if } k+r=t_{m+n+1}+s,\ m+n+1=f_{\mu+\nu}+t, \\ &= \rho(\theta_1^{(m+n)e})g, \text{ if } k+r=t_{m+n+1}+s,\ m+n+1=f_{\mu+\nu+1}+t, \\ &= \rho(\theta_1^{(m+n)e})g^2, \text{ if } k+r=t_{m+n+1}+s,\ m+n+1=f_{\mu+\nu+2}+t. \end{aligned}$$

Now, since $j_pj_e=\beta_ej_ej_p$, we have

$$c_{bp,ne} = \beta_{ne}\beta_{ne}(\theta_p)\cdots\beta_{ne}(\theta_p^{b-1}), \tag{35}$$

and by (10) and (11)

$$
(36)\qquad \begin{aligned}
\beta_{re} &= \beta_e\,\beta_{(r-1)}(\theta_1^e) && (r \neq f_\nu),\\
\beta_{re} &= \frac{g}{g(\theta_p)}\,\beta_e\,\beta_{(r-1)e}(\theta_1^e) && (r = f_\nu).
\end{aligned}
$$

For $ex \equiv e \pmod{p}$ and accordingly $c_{e,(r-1)e} = c_{e_0,(r-1)e_0}$.

Hence, by (17), the second of (28), (35), and (36),

$$
(37)\qquad c_{bp,ne} = \frac{G_n}{G_n(\theta_p^b)}\left(\frac{g}{g(\theta_p^b)}\right)^\nu,
$$

where $G_n = \rho\,\rho(\theta_e) \cdots \rho(\theta_e^{n-1})$, and $n = f_\nu + w$.

When $k+r<P$, $c_{kp,rp}=1$ and $u=(k+r)p$, and if we take $k=1$, D_2 by means of (34) and (37) becomes

$$
(38)\qquad Y\alpha_p\alpha_{rp}(\theta_p^s)\,\frac{G_n}{G_n(\theta_p^s)}\left(\frac{g}{g(\theta_p^s)}\right)^\nu = \alpha_{(r+1)p}
$$

where

$$
\begin{aligned}
Y &= 1, && r \neq t_{n+1} - 1,\\
&= \rho(\theta_e^{n+m}), && r+1 = t_{n+1}, \quad n+1 \neq f_{\nu+1},\\
&= g\rho(\theta_e^{n+m}), && r+1 = t_{n+1}, \quad n+1 = f_{\nu+1}.
\end{aligned}
$$

From successive applications of (38) we get*

$$
(39)\qquad \alpha_{rp} = \alpha_p\alpha_p(\theta_p^s) \cdots \alpha_p(\theta_p^{(r-1)s})\rho\rho(\theta_e) \cdots \rho(\theta_e^{n-1})g^\nu,
$$

where $r=1, 2, \cdots, P-1$; $r=t_n+v$; $n=f_\nu+w$.

By means of (34) and the formula $\theta_p^b\theta_1^{me}=\theta_p^{ks}=\theta_{kp_0}$, it can be shown that D_2 is satisfied identically when the values of α_{kp}, α_{rp} and α_u are substituted from (39) into (32), for all values of k and r for which $k+r<P$.

But, if $k+r=P$, $c_{kp,rp}=\rho$ and $u=e$. Hence

$$
(k+r)z = Pz, \quad k+r = t_s,
$$

and, since $kz \not\equiv 0 \pmod{P}$ $(k \leqq P-1)$, $z=m+n+1$. If

$$
(40)\qquad z = f_\lambda + h \qquad\qquad (a_\lambda + he < p),
$$

$\lambda=\mu+\nu$ or $\mu+\nu+1$ or $\mu+\nu+2$, and in all cases by (34) and (39) D_2 reduces to†

$$
(41)\qquad \alpha_p\alpha_p(\theta_p^s) \cdots \alpha_p(\theta_p^{s(P-1)})\rho\rho(\theta_e) \cdots \rho(\theta_e^{s-1})g^\lambda = \rho(\theta_q)\alpha_e.
$$

---

* If $e=0$, $\nu\equiv 0$ and $\alpha_{rp}\equiv\alpha_p\alpha_p(\theta_p^s) \cdots \alpha_p(\theta_p^{(r-1)s})\rho^n$.

† If $e=0$, $\lambda=0$, $\alpha_e=1$ and (41) becomes $\alpha_p\alpha_p(\theta_p^s) \cdots \alpha_p(\theta_p^{s(P-1)})\,\rho^s=\rho(\theta_q)$.

Similarly, if $k+r>P$, D_2 reduces to (41) for all values of $k<P$, $r<P$. For, when $k+r>P$, $c_{kp,rp}=\rho$ and $u=e+(k+r-P)p$. Now

$$\alpha_e\alpha_{(k+r-P)p}(\theta_e^y)c_{e_0,(k+r-P)p_0} = \alpha_{e+(k+r-P)p},$$

and by (26) D_2 becomes

$$\alpha_{kp}\alpha_{rp}(\theta_p^{ks})c_{kp_0,rp_0} = \rho(\theta_q)c_{es,(k+r-P)sp}\alpha_e\alpha_{(k+r-P)p}(\theta_p^{Ps}), \tag{42}$$

and, if

$$k + r = t_s + a \qquad (a_s + as < P),$$

then

$$z(k + r) = sP + a_s + az, \quad k + r - P = t_{s-s} + a\,.$$

Hence, if $s=f_\sigma+n$, where $a_\sigma+ne<p$, the left hand side of (42) is equal to

$$\alpha_p\alpha_p(\theta_p^s) \cdots \alpha_p(\theta_p^{(k+r-1)s})\rho\rho(\theta_e) \cdots \rho(\theta_e^{e-1})g^\sigma.$$

Then, if

$$s - z = f_\mu + n' \qquad (a_\mu + n'e < p),$$

by (40)

$$s = f_{\lambda+\mu} + n'' \text{ or } f_{\lambda+\mu+1} + n'',$$

and so $\sigma=\lambda+\mu$ or $\lambda+\mu+1$, according as $c_{es,(k+r-P)s}=1$ or g. The right hand side of (42) then becomes

$$\rho(\theta_q)\alpha_e\alpha_p(\theta_p^{Ps})\alpha_p(\theta_p^{(P+1)s}) \cdots \alpha_p(\theta_p^{(k+r-1)s})X,$$

where

$$X = \rho(\theta_e^z)\rho(\theta_e^{z+1}) \cdots \rho(\theta_e^{e-1})g^{\sigma-\lambda}.$$

On equating the two sides so obtained and cancelling the common factors, we get (41).

We must now consider the general case of D_2, where

$$\begin{aligned} k &= a + bp \\ r &= t + sp \end{aligned} \qquad \begin{pmatrix} a,\ t = 1,2, \cdots ,p - 1 \\ b,\ s = 1,2, \cdots ,P - 1 \end{pmatrix}.$$

For simplicity in writing let

$j_1^{a'}$ be defined as j_a when $\Theta_1^{a'} = \Theta_a$ and $a' > p > a$,

$j_p^{b'}$ be defined as j_{bp+d} when $\Theta_p^{b'} = \Theta_{bp+d}$ and $b' > P > b$.

Then

$$j_1^a j_p^b j_1^t j_p^s = c_{bt}(\theta_1^a) j_1^a j_1^{tz^b} j_p^b j_p^s,$$

$$j_k j_r = c_{bt}(\theta_1^a) c_{a,tz^b} j_v c_{bp,sp} j_w.$$

Hence

$$c_{kr} = c_{bt}(\theta_1^a)c_{a,tz^b}c_{bp,sp}(\theta_1^v)c_{vw}. \tag{43}$$

To get the value of $c_{k_0r_0}$ we consider*

$$j_1^{ay}j_p^{bz}j_1^{ty}j_p^{sz}$$

which is equal to

$$c_{ay,bzp}j_{k_0}c_{ty,szp}j_{r_0} = c_{ay,bzp}c_{ty,szp}(\theta_{k_0})c_{k_0r_0}j_{u_0}. \tag{44}$$

Since j_{bp_0} may be of the form $j_1^n j_p^m$ we have

$$c_{tyx^{bz},bzp}j_p^{bz}j_1^{ty} = c_{bzp,ty}j_1^{tyx^{bz}}j_p^{bz},$$

or, since $x^s \equiv x \pmod{p}$,

$$c_{tyx^b,bzp}j_p^{bz}j_1^{ty} = c_{bzp,ty}j_1^{tyx^b}j_p^{bz}. \tag{45}$$

Hence

$$c_{tyx^b,bzp}(\theta_1^{ay})j_1^{ay}j_p^{bz}j_1^{ty}j_p^{sz} \tag{46}$$

$$= c_{bzp,ty}(\theta_1^{ay})c_{ay,tyx^b}c_{bzp,szp}(\theta_{v_0})c_{v_0w_0}j_{u_0},$$

where

$$j_1^{ay}j_1^{tyx^b} = c_{ay,tyx^b}j_{v_0},$$

$$j_p^{bz}j_p^{sz} = c_{bzp,szp}j_{w_0}.$$

We get as special cases of D_2,

$$\alpha_v\alpha_w(\theta_{v_0})c_{v_0w_0} = c_{vw}(\theta_q)\alpha_u,$$

$$\alpha_a\alpha_{tx^b}(\theta_{a_0})c_{ay,tx^by} = c_{a,tx^b}(\theta_q)\alpha_v, \tag{47}$$

$$\alpha_{bp}\alpha_{sp}(\theta_{bp_0})c_{bzp,szp} = c_{bp,sp}(\theta_q)\alpha_w,$$

and

$$\alpha_k = \alpha_{a+bp} = \alpha_a\alpha_{bp}(\theta_{a_0})c_{ay,bzp} \tag{48}$$

$$(a = 0,1,\cdots,p-1\,;\; b = 0,1,\cdots,P-1),$$

where (48) combined with (30) and (39) defines α_k in terms of α and α_p, and $c_{ay,bzp} = 1$ or g according as $a_m + s_\mu < p$ or $\geqq p$, where

$$ay = mp + a_m \quad (a_m < p), \quad bz = sP + b_s \quad (b_s < P), \quad se = \mu p + s_\mu \quad (s_\mu < p).$$

---

* If $e=0$, $c_{ny,mzp}=1$ for all values of n and m.

Making use of (47) and (48), and substituting for c_{kr} and $c_{k_0r_0}$ their values obtained from (44), (45) and (46) in D_2, we get

$$\alpha_{bp}\alpha_t(\theta_p^{bs})c_{bsp,ty} = c_{tyx^b,bsp}c_{bp,t}(\theta_q)\alpha_{tx^b}\alpha_{bp}(\theta_1^{tyx^b}). \tag{49}$$

The w in the first of (47) may be of the form $t+sp$ and so the first of (47) is a case of D_2 that we are considering. But by writing $a=v$, $b=0$, and proceeding as in the general case, we reduce it to (49), where since $b=0$ the formula corresponding to the first of (47) is now of the type (48). The second and third of (47) have been treated earlier.

We now prove the following lemma:

Lemma A. *The formula* (49) *may be deduced for all values of $b<P$ and $t<p$ from*

$$\alpha_p\alpha(\theta_p^s)c_{sp,y} = c_{yx,sp}c_{p,1}(\theta_q)\alpha_x\alpha_p(\theta_1^{xy}). \tag{50}$$

Assume that (49) holds for all values of $b\leqq b$ and $t\leqq t$, and consider (49) with $t=1$; that is

$$\alpha_{bp}\alpha(\theta_p^{bs})c_{bsp,y} = c_{yx^b,bsp}c_{bp,1}(\theta_q)\alpha_{x^b}\alpha_{bp}(\theta_1^{yx^b}). \tag{51}$$

If we now write $\theta_1^{tyx^b}$ for i in (51) and multiply the left members of (51) and (49) together and equate the result to the product of the right members, we get

$$\begin{aligned}\alpha_{bp}\alpha_{t+1}(\theta_p^{bs})c_{bsp,ty}c_{bsp,y}(\theta_1^{tyx^b})c_{tx^by,yx^b}\\ = Y\alpha_{(t+1)x^b}\alpha_{bp}(\theta_1^{(t+1)yx^b})\end{aligned} \tag{52}$$

where

$$Y = c_{ty,y}(\theta_p^{bs})c_{bp,1}(\theta_1^{x^b}\theta_q)c_{tyx^b,bpx}c_{yx^b,bsp}(\theta_1^{tyx^b})c_{tx^b,x^b}(\theta_q).$$

Now,

$$\begin{aligned}c_{ty,y}(\theta_p^{bs})c_{bsp,(t+1)y}c_{tyx^b,bsp}c_{yx^b,bsp}(\theta_1^{tyx^b})\\ = c_{bsp,ty}c_{bsp,y}(\theta_1^{tyx^b})c_{tyx^b,yx^b}c_{(t+1)yx^b,bsp},\end{aligned}$$

and

$$c_{bp,t+1} = c_{bp,1}(\theta_1^{tx^b})c_{bp,t}c_{tx^b,x^b}.$$

Making use of these two results, we see that (52) becomes (49) with t replaced by $t+1$, and so by induction (49) may be deduced from (51).

Now, (49) with $t=x$ becomes

$$\alpha_{bp}\alpha_x(\theta_p^{bs})c_{bsp,xy} = c_{yx^{b+1},bsp}c_{bp,x}(\theta_q)T, \tag{53}$$

where

$$T = \alpha_{x^{b+1}}\alpha_{bp}(\theta_1^{yx^{b+1}}).$$

Since

$$c_{(b+1)p,1} = c_{p,1}(\theta_p{}^b)c_{bp,z}$$

and

$$c_{bzp,zp}\,c_{(b+1)zp,y}\,c_{yz,zp}(\theta_p{}^{bz})c_{yz^{b+1},bpz}$$
$$= c_{zp,y}(\theta_p{}^{bz})c_{bzp,zy}\,c_{bzp,zp}(\theta_1{}^{yz^{b+1}})c_{yz^{b+1},(b+1)zp},$$

when we combine (53) with (50), where θ_p^{bz} is written for i in (50), we get (49) with b replaced by $b+1$ and our lemma is proved. Since $z<P$, $c_{yz,zp}=1$ and (50) becomes

(54) $$\alpha_p\alpha(\theta_p{}^z)c_{zp,y} = c_{p,1}(\theta_q)\alpha_z\alpha_p(\theta_1{}^{zy}),$$

where

$$c_{p1} = \beta,\ c_{zp,y} = \beta_y(\theta_p{}^{z-1})\beta_{yz}(\theta_p{}^{z-2})\cdots\beta_{yz^{z-1}}.$$

We have now shown that the condition D_2 reduces for all values of $k<q$, $r<q$ to (30), (31), (39), (41), (48), and (54) where (30), (39), and (48) merely express $\alpha_k(k<q)$ in terms of α and α_p.

It remains to consider the condition D_3. If $j_{e'}j_k=d_kj_{k'}j_{e'}$, where $j_{k'}=j_{k_0}\cdots$, and there are Q subscripts 0, $k'=a'+b'p$, where $a'=ay^Q\equiv ax^{e_1}$ (mod p) by (26), and $b'=bz^Q=bmP+b$ by (24), and accordingly

$$j_p{}^{b'} = j_1{}^{me}j_p{}^b.$$

Also $c_{e'k}=d_kc_{k'e'}$ and D_3 becomes

(55) $$c_{e'k}\delta = c_{k'e'}\alpha_{a+bp}(\theta_q{}^{Q-1})\cdots\alpha_{ay^{Q-1}+bz^{Q-1}p}\delta(\theta_{k'}).$$

We shall now prove the following lemma:

Lemma B. *Condition D_3 follows for all values of $k<q$ from* (56) *and* (57):

(56) $$c_{e',1}\delta = c_{x^{e_1},e'}\alpha(\theta_q{}^{Q-1})\alpha_y(\theta_q{}^{Q-2})\cdots\alpha_{y^{Q-1}}\delta(\theta_1{}^{x^{e_1}}),$$

(57) $$c_{e',p}\delta = c_{z^Qp,e'}\alpha_p(\theta_q{}^{Q-1})\alpha_{zp}(\theta_q{}^{Q-2})\cdots\alpha_{z^{Q-1}p}\delta(\theta_p{}^{z^Q}).$$

Since (55) holds for all values of $k<q$, it is true in particular for the two cases $b=0$ and $a=0$ respectively:

(58) $$c_{e',a}\delta = c_{ax^{e_1},e'}\alpha_a(\theta_q{}^{Q-1})\alpha_{ay}(\theta_q{}^{Q-2})\cdots\alpha_{ay^{Q-1}}\delta(\theta_1{}^{ax^{e_1}}),$$

(59) $$c_{e',bp}\delta = c_{b'p,e'}\alpha_{bp}(\theta_q{}^{Q-1})\alpha_{bzp}(\theta_q{}^{Q-2})\cdots\alpha_{bz^{Q-1}p}\delta(\theta_p{}^{b'}).$$

If we write

$$\theta_1^{ay^Q} = \theta_1^{ax^{e_1}}$$

for i in (59), since

$$\theta_1^{ay^s}\theta_q{}^{Q-s} = \theta_q{}^{Q-s}\theta_1^{ay^Q},$$

we have from (58) and (59)

$$c_{e',a}c_{e',bp}(\theta_1^{ay^Q})\delta = c_{ax^{e_1},e'}c_{b'p,e'}(\theta_1^{ax^{e_1}})\delta(\theta_p{}^{b'}\theta_1^{ax^{e_1}})X, \tag{60}$$

where

$$X = \prod_{s=1}^{s=Q} \frac{\alpha_{ay^{s-1}+bz^{s-1}p}(\theta_q{}^{Q-s})c_{ay^{s-1},bz^{s-1}p}(\theta_q{}^{Q-s+1})}{c_{ay^s,bz^sp}(\theta_q{}^{Q-s})}$$

$$= c_{a,bp}(\theta_q{}^Q)\left[c_{ay^Q,bz^Qp}\right]^{-1}\prod_{s=1}^{s=Q}\alpha_{ay^{s-1}+bz^{s-1}p}(\theta_q{}^{Q-s}).$$

Now, since $meb+e_1(x^b-1)\equiv 0 \pmod{p}$ by (24),

$$j_p^{b'}j_{e'} = j_1^{meb}j_p^bj_1^{e_1}j_p^{e_1} = fj_1^{e_1}j_p^bj_p^{e_1} = fj_{e'}j_p^b \qquad (f \neq 0 \text{ and in } F(i)).$$

Hence,

$$j_1^{ax^{e_1}}j_p^{b'}j_{e'} = c_{ax^{e_1},b'p}c_{k',e'}j_u$$

$$= \frac{c_{b'p,e'}(\theta_1{}^{ax^{e_1}})c_{ax^{e_1},e'}c_{a,bp}(\theta_{e'})c_{e'k}j_u}{c_{e',bp}(\theta_1{}^{ax^{e_1}})c_{e'a}}.$$

From this result remembering that $ax^{e_1}\equiv ay^Q \pmod{p}$ and that $\Theta_p^Q=\Theta_{e'}$, we see that (60) becomes (55). By induction, in a manner similar to that used in Lemma B of §4, it can be shown that (58) and (59) are consequences of (56) and (57) respectively. In the proof we require the formulas

$$c_{e',a+1}c_{ax^{e_1},e'}c_{x^{e_1},e'}(\theta_1{}^{ax^{e_1}}) = c_{e'a}c_{e',1}(\theta_1{}^{ax^{e_1}})c_{ax^{e_1},x^{e_1}}c_{(a+1)x^{e_1},e'},$$

$$c_{bp,p}(\theta_{e'})c_{e',(b+1)p}c_{b'p,e'}c_{zQp,e'}(\theta_p{}^{b'})$$

$$= c_{e',bp}c_{e'p}(\theta_p{}^{b'})c_{b'p,zQp}c_{(b+1)'p,e'},$$

which can be deduced as in the previous cases. Since

$$c_{e'1} = c_{e_2p,1}(\theta_1{}^{e_1})c_{e_1,x^{e_1}} \text{ and } c_{e_1,x^{e_1}} = c_{x^{e_1},e_1} = c_{x^{e_1},e'},$$

(56) becomes

$$c_{e_2p,1}(\theta_1{}^{e_1})\delta = \alpha(\theta_q{}^{Q-1})\alpha_y(\theta_q{}^{Q-2})\cdots\alpha_{y^{Q-1}}\delta(\theta_1{}^{x^{e_1}}). \tag{61}$$

But $e_2\neq P-1$ by (26) and so $c_{e'p}=1$ and (57) becomes

$$\delta = c_{zQp,e'}\alpha_p(\theta_q{}^{Q-1})\alpha_{zp}(\theta_q{}^{Q-2})\cdots\alpha_{z^{Q-1}p}\delta(\theta_p{}^{z^Q}). \tag{62}$$

In (61)

$$c_{e_2p,1} = \beta(\theta_p^{e_2-1})\beta_z(\theta_p^{e_2-2})\beta_{z^2}(\theta_p^{e_2-3}) \cdots \beta_{z^{e_2-1}},$$

and in (62), since $z^Q = mP+1$,

$$c_{zQp,e'} = \beta_{e'}(\theta_1^{me})c_{me,e_1z},$$

where $c_{me,e_1z}=1$ or g, according as $t \leqq e_1$ or $>e_1$ and $e_1x \equiv t \pmod{p}$.

We have now proved

THEOREM B. *Let $f(x)=0$ be an equation of degree $n=QPp$, irreducible in a field F, whose group for F is generated by three generators Θ_1, Θ_p and Θ_q described in* §5. *Then the algebra Σ is associative if and only if conditions* (28) *hold. The totality of polynomials in j_q with coefficients in Σ form an algebra Γ of order n^2 over F which is associative if and only if conditions* (29), (31), (41), (54), (61), *and* (62) *all hold and Σ is associative.*

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A GENERALIZATION OF TAYLOR'S SERIES*

BY

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1. **Introduction.** In view of the great importance of Taylor's series in analysis, it may be regarded as extremely surprising that so few attempts at generalization have been made. The problem of the representation of an arbitrary function by means of linear combinations of prescribed functions has received no small amount of attention. It is well known that one phase of this problem leads directly to Taylor's series, the prescribed functions in this case being polynomials. It is the purpose of the present paper to discuss this same phase of the problem when the prescribed functions are of a more general nature.

Denote the prescribed functions by

$$u_0(x), \quad u_1(x), \quad u_2(x), \cdots, \tag{1}$$

real functions of the real variable all defined in a common interval $a \leqq x \leqq b$. Set

$$s_n(x) = c_0 u_0(x) + c_1 u_1(x) + \cdots + c_n u_n(x).$$

It is required to determine the constants c_i in such a way that $s_n(x)$ shall be the best approximation to a given function $f(x)$ that can be obtained by a linear combination of $u_0, u_1, \cdots, u_n$. Of course this problem becomes definite only after a precise definition of the phrase "best approximation" has been given. Various methods have been used, of which we mention the following:

(A) The method of least squares;
(B) The method of Tchebycheff;
(C) The method of Taylor.

In each of these cases the functions (1) may be so restricted that the constants c_i are uniquely determined. The function $s_n(x)$ thereby determined is called a *function of approximation*. Having determined the functions of approximation, one is led directly to an expansion problem. Under what conditions will $s_n(x)$ approach $f(x)$ as n becomes infinite? Or, when will the series

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(2) $$s_0(x) + [s_1(x) - s_0(x)] + [s_2(x) - s_1(x)] + \cdots$$

converge and represent $f(x)$ in (a, b) or in any part of (a, b)?

There are two special sequences (1) that have received particular attention:

(1′) $$1, x, x^2, \cdots,$$

(1″) $$1, \sin x, \cos x, \sin 2x, \cos 2x, \cdots.$$

The following scheme will serve as a partial reference list to this field, and to put into evidence the gap in the general theory which it is hoped the present paper will in some measure fill.

	(A)*	(B)*	(C)
(1′)	A. M. Legendre	P. L. Tchebycheff	B. Taylor
(1″)	J. J. Fourier	M. Fréchet	G. Teixeira†
(1)	E. Schmidt	A. Haar	

The entry in the upper left-hand corner, for example, means that the series (2) becomes for the method (A) and for the special sequence (1′) the expansion of $f(x)$ in a series of Legendre polynomials. It should be pointed out that the series studied by Teixeira were considered by him in another connection, and that no mention of their relation to Taylor's series was made.

It is found that if certain restrictions are imposed on the sequence (1), and if the functions of approximation are determined according to the method (C), then the general term of the series (2) may be factored, just as in Taylor's series, into two parts $c_n g_n(x)$, the second of which depends in no way on the function $f(x)$ represented, the constant c_n alone being altered when $f(x)$ is altered. As in the case of Taylor's series the constant c_n is determined by means of a linear differential operator of order n. If further restrictions, Conditions A of §6, are imposed on the sequence (1), it is found that series (2) possesses many of the formal properties of a power series. If t is a point at which $s_n(x)$ has closest contact with $f(x)$, then the interval of convergence of (2) extends equal distances on either side of t (provided that the interval of definition (a, b) permits). The familiar process of analytic extension also applies to this generalized power series.

A necessary and sufficient condition for the representation of a function $f(x)$ is obtained by generalizing a theorem of S. Bernstein. Then imposing

* For references, see Encyklopädie der Mathematischen Wissenschaften, IIC9c (Fréchet-Rosenthal), §51.

† *Extrait d'une lettre de M. Gomes Teixeira à M. Hermite*, Bulletin des Sciences Mathématiques et Astronomiques, vol. 25 (1890), p. 200.

further conditions, Conditions B of §10, it is found possible to represent an arbitrary analytic function in a series (2). It is shown that the conditions are not so strong as to exclude the case of Taylor's series, and that sequences (1) exist, satisfying the conditions, and leading to series quite different from Taylor's series. Finally the relation of the general series to Teixeira's series is shown.

2. **The Taylor method of approximation and the existence of the functions of approximation.** The Taylor method of approximation consists in determining the constants c_i of $s_n(x)$ in such a way that the approximation to $f(x)$ shall be as close as possible in the immediate neighborhood of a point $x=t$ of (a, b), irrespective of the magnitude of the error $|f(x)-s_n(x)|$ at points x remote from t. More precisely, the constants c_i are determined so that the curves $y=f(x)$ and $y=s_n(x)$ shall have closest contact at a point t. If the functions $f(x)$ and $s_n(x)$ are of class C^{m+1}(possess continuous derivatives of order $m+1$) in the neighborhood of $x=t$, then the curves $y=f(x)$ and $y=s_n(x)$ (or the functions $f(x)$ and $s_n(x)$ themselves) are said to have contact of order m at $x=t$ if and only if

$$f^{(k)}(t) = s_n^{(k)}(t), \quad k = 0,1, \cdots ,m ; \quad f^{(m+1)}(t) \neq s_n^{(m+1)}(t).$$

We now make the following

DEFINITION. *The function*

$$s_n(x) = \sum_{i=0}^{n} c_i u_i(x)$$

is a function of approximation of order n for the point $x=t$ if the functions $u_i(x)$ are of class C^n in the neighborhood of $x=t$, and if $s_n(x)$ has contact of order n at least with $f(x)$ at $x=t$.

We shall have frequent occasion to use Wronskians, so that it will be convenient to introduce a notation. The functions $v_0(x), v_1(x), \cdots, v_n(x)$ being of class C^n, we set

$$W[v_0(x),v_1(x), \cdots ,v_n(x)] = \begin{vmatrix} v_0(x) & v_1(x) & \cdots v_n(x) \\ v_0'(x) & v_1'(x) & \cdots v_n'(x) \\ \cdot & \cdot & \cdots \cdot \\ \cdot & \cdot & \cdots \cdot \\ v_0^{(n)}(x) & v_1^{(n)}(x) & \cdots v_n^{(n)}(x) \end{vmatrix}.$$

In particular, for the functions of the sequence (1) we set

$$W_n(x) = W[u_0(x),u_1(x), \cdots ,u_n(x)].$$

We may now state

THEOREM I. *If the functions $f(x)$, $u_0(x)$, $u_1(x)$, $\cdots$, $u_n(x)$ are of class C^n in the neighborhood of $x=t$, and if $W_n(t)\neq 0$, then there exists a unique function of approximation*

$$s_n(x) = \sum_{i=0}^{n} c_i u_i(x) = -\left(\frac{1}{W_n(t)}\right)\begin{vmatrix} 0 & u_0(x) & u_1(x) & \cdots & u_n(x) \\ f(t) & u_0(t) & u_1(t) & \cdots & u_n(t) \\ f'(t) & u_0'(t) & u_1'(t) & \cdots & u_n'(t) \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ f^{(n)}(t) & u_0^{(n)}(t) & u_1^{(n)}(t) & \cdots & u_n^{(n)}(t) \end{vmatrix}$$

of order n for $x=t$.

The proof of this theorem consists in noting that the determinant of the system of equations

$$f^{(k)}(t) = c_0 u_0^{(k)}(t) + c_1 u_1^{(k)}(t) + \cdots + c_n u_n^{(k)}(t) \qquad (k = 0, 1, \cdots, n)$$

is $W_n(t)$, which is different from zero by hypothesis, and in solving the system for the constants c_i. The values of the c_i thus obtained give the above expression for $s_n(x)$.

3. **Determination of the form of the series.** In order to form the series (2) we need to know the existence of the functions of approximation of all orders. We shall assume then that $f(x)$ and $u_i(x)$, $i=0, 1, 2, \cdots$, are of class C^∞ in the interval $a\leqq x\leqq b$. Moreover we shall assume* that $W_i(x)>0$ in the same interval. This insures the existence of the functions of approximation of all orders for an arbitrary point of the interval. We are thus led naturally to a set of functions (1) possessing what G. Pólya† has called the Property W.

DEFINITION. *The sequence (1) is said to possess the Property W in (a, b) if each function of the sequence is of class C^∞ in $a\leqq x\leqq b$, and if $W_i(x)>0$, $i=0, 1, 2, \cdots$, in the same interval.*

We shall now be able to show that the series (2) has the form

$$c_0 h_0(x) + c_1 h_1(x) + c_2 h_2(x) + \cdots,$$

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* No gain in generality would be obtained by allowing some or all of the functions $W_i(x)$ to be negative.

† G. Pólya, *On the mean-value theorem corresponding to a given linear homogeneous differential equation*, these Transactions, vol. 24 (1922), p. 312. We have extended the definition to apply to an infinite set.

where the functions $h_i(x)$ depend only on the sequence (1) and on the choice of the point t, and not at all on the function $f(x)$ to be expanded. The constants c_i, on the other hand, are independent of x, but depend on the function $f(x)$ and on the choice of the point t. It is this property of the series (2) that makes all the series under the method (A) of the introduction so convenient to use. The property is lacking for the method (B), and for this reason the Tchebycheff series are less useful in spite of their theoretical advantages.

The direct factorization of $[s_n(x)-s_{n-1}(x)]$ is attended with algebraic difficulties which may be avoided by means of the following device. Set

$$\phi(x) = s_n(x) - s_{n-1}(x).$$

Then by Theorem I

$$\overset{(k)}{s_n}(t) = f^{(k)}(t) = \overset{(k)}{s_{n-1}}(t), \quad k = 0,1, \cdots, n-1, \quad s_n^{(n)}(t) = f^{(n)}(t),$$

$$\phi^{(k)}(t) = 0, \quad k = 0,1, \cdots, n-1, \quad \phi^{(n)}(t) = f^{(n)}(t) - \overset{(n)}{s_{n-1}}(t).$$

But $\phi(x)$ by its form is a linear combination of $u_0(x), u_1(x), \cdots, u_n(x)$,

$$\phi(x) = a_0u_0(x) + a_1u_1(x) + \cdots + a_nu_n(x).$$

Hence the constants a_i must satisfy the equations

$$0 = a_0u_0^{(k)}(t) + a_1u_1^{(k)}(t) + \cdots + a_nu_n^{(k)}(t) \qquad (k = 0,1, \cdots, n-1),$$

$$f^{(n)}(t) - \overset{(n)}{s_{n-1}}(t) = a_0u_0^{(n)}(t) + a_1u_1^{(n)}(t) + \cdots + a_nu_n^{(n)}(t).$$

From these equations we see that $\phi(x)$ must satisfy the equation

$$\phi(x)W_n(t) = [f^{(n)}(t) - \overset{(n)}{s_{n-1}}(t)] \begin{vmatrix} u_0(t) & u_1(t) & \cdots & u_n(t) \\ u_0'(t) & u_1'(t) & \cdots & u_n'(t) \\ \cdot & \cdot & \cdots & \cdot \\ u_0^{(n-1)}(t) & u_1^{(n-1)}(t) & \cdots & u_n^{(n-1)}(t) \\ u_0(x) & u_1(x) & \cdots & u_n(x) \end{vmatrix}.$$

The factorizaton of $\phi(x)$ which we set out to perform is thus completed. For brevity we set

$$(3) \qquad g_n(x,t) = \left(\frac{1}{W_n(t)}\right) \begin{vmatrix} u_0(t) & u_1(t) & \cdots & u_n(t) \\ u_0'(t) & u_1'(t) & \cdots & u_n'(t) \\ \cdot & \cdot & \cdots & \cdot \\ u_0^{(n-1)}(t) & u_1^{(n-1)}(t) & \cdots & u_n^{(n-1)}(t) \\ u_0(x) & u_1(x) & \cdots & u_n(x) \end{vmatrix},$$

so that $g_n(x, t)$ is the function $h_n(x)$ sought. For convenience in later work we have put into evidence the point t chosen. We see that

$$s_n(x) - s_{n-1}(x) = [f^{(n)}(t) - s_{n-1}^{(n)}(t)]g_n(x,t). \tag{4}$$

Now by reference to the explicit form of $s_{n-1}(x)$ given in Theorem I it becomes clear that

$$f^{(n)}(t) - s_{n-1}^{(n)}(t)$$
$$= f^{(n)}(t) + \left(\frac{1}{W_{n-1}(t)}\right) \begin{vmatrix} 0 & u_0^{(n)}(t) & u_1^{(n)}(t) & \cdots & u_{n-1}^{(n)}(t) \\ f(t) & u_0(t) & u_1(t) & \cdots & u_{n-1}(t) \\ f'(t) & u_0'(t) & u_1'(t) & \cdots & u_{n-1}'(t) \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ f^{(n-1)}(t) & u_0^{(n-1)}(t) & u_1^{(n-1)}(t) & \cdots & u_{n-1}^{(n-1)}(t) \end{vmatrix}$$

$$= \frac{W[u_0(t), u_1(t), \cdots, u_{n-1}(t), f(t)]}{W_{n-1}(t)}.$$

It will now be convenient to introduce a linear differential operator defined by the relation

$$L_n f(x) = \frac{W[u_0(x), u_1(x), \cdots, u_{n-1}(x), f(x)]}{W_{n-1}(x)}.$$

By use of this notation equation (4) becomes

$$s_n(x) - s_{n-1}(x) = L_n f(t) g_n(x,t),$$

and the expansion of the function $f(x)$ has the form

$$f(x) \sim L_0 f(t) g_0(x,t) + L_1 f(t) g_1(x,t) + L_2 f(t) g_2(x,t) + \cdots, \tag{5}$$

$$g_0(x,t) = \frac{u_0(x)}{u_0(t)}, \qquad L_0 f(x) = f(x).$$

Incidentally, we have proved the following formula:

$$L_n f(t) = f^{(n)}(t) - L_0 f(t) g_0^{(n)}(t,t) - L_1 f(t) g_1^{(n)}(t,t) - \cdots$$
$$- L_{n-1} f(t) g_{n-1}^{(n)}(t,t).$$

4. **The properties of the functions $g_n(x, t)$ and of the operators L_n.** From the equation (3) defining the function $g_n(x, t)$ we read off at once certain properties. Considered as a function of x, it is evidently a linear combination of $u_0(x), u_1(x), \cdots, u_n(x)$ satisfying the equations

$$\frac{\partial^k}{\partial x^k} g_n(x,t)\Big|_{x=t} = \begin{cases} 0, & k = 0,1,\cdots,n-1, \\ 1, & k = n. \end{cases} \tag{6}$$

The operator L_n is seen to be a linear differential operator of order n which annuls the first n functions of the set (1), and which satisfies the relation

$$L_n x^n \big|_{x=0} = n!.$$

The expanded form of $L_n f$ is

$$L_n f(x) = f^{(n)}(x) + p_1(x)f^{(n-1)}(x) + \cdots + p_n(x)f(x),$$

the coefficient of $f^{(n)}(x)$ being unity.

The function $g_n(x, t)$ is the function of Cauchy* used in obtaining a particular solution of the non-homogeneous equation

$$L_{n+1}f(x) = p(x)$$

from the solutions of the corresponding homogeneous equation. The particular solution of this equation vanishing with its first n derivatives at $x=t$ is known to be

$$f(x) = \int_t^x g_n(x,t)p(t)\,dt.$$

When L_n operates on the functions $g_m(x, t)$ the result is particularly simple. Since L_n annuls the first n functions of the sequence (1), it follows that

$$L_n g_m(x,t) \equiv 0, \quad m < n.$$

Let us also compute $L_n g_m(x, t)$ for $x=t$ and $m \geqq n$. By means of the relations (6) we find that

$$L_n g_n(x,t)\big|_{x=t} = \frac{\partial^n}{\partial x^n} g_n(x,t)\big|_{x=t} + \left[p_1(x)\frac{\partial^{n-1}}{\partial x^{n-1}} g_n(x,t)\right]_{x=t} + \cdots = 1,$$

$$L_n g_{n+p}(x,t)\big|_{x=t} = \frac{\partial^n}{\partial x^n} g_{n+p}(x,t)\big|_{x=t} + \left[p_1(x)\frac{\partial^{n-1}}{\partial x^{n-1}} g_{n+p}(x,t)\right]_{x=t} + \cdots = 0$$

$$(p = 1,2,\cdots).$$

These properties† may be summed up as follows:

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* E. Goursat, *Cours d'Analyse Mathématique*, vol. 2, p. 430.

† An a priori discussion of the series in question might be made by starting with these formulas. They may evidently be used to determine the coefficients of the series formally.

$$L_n g_m(x,t)\big|_{x=t} = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases} \tag{7}$$

The relation of the series (5) to Taylor's series is brought out more clearly if the sequence (1) is replaced by the sequence (1′) in the preceding work. Simple computations show that for this case

$$W_n(x) = n!(n-1)!(n-2)!\cdots 2!,$$

$$L_n f(x) = f^{(n)}(x),$$

$$g_n(x,t) = (x-t)^n/n!.$$

The series (5) now has precisely the form of Taylor's series.

For many purposes it will be convenient to use another form of the differential operator L_n. It is known* that if the Property W holds for the set $u_0(x), u_1(x), \cdots, u_{n-1}(x)$ in (a, b), then $L_n f(x)$ may be written as

$$L_n f(x) = \phi_0(x)\phi_1(x)\cdots\phi_{n-1}(x)\frac{d}{dx}\frac{1}{\phi_{n-1}(x)}\frac{d}{dx}\frac{1}{\phi_{n-2}(x)}\frac{d}{dx}\cdots\frac{d}{dx}\frac{1}{\phi_1(x)}\frac{d}{dx}\frac{f(x)}{\phi_0(x)}, \tag{8}$$

where

$$\phi_0(x) = W_0(x), \quad \phi_1(x) = \frac{W_1(x)}{[W_0(x)]^2}, \quad \phi_k(x) = \frac{W_k(x)W_{k-2}(x)}{[W_{k-1}(x)]^2}$$

$$(k = 2, 3, \cdots, n-1).$$

The functions $\phi_i(x)$ will all be positive for $a \leqq x \leqq b$ since we are assuming that the Property W holds in that interval. The differential expression adjoint to $L_n f(x)$ may then be written†

$$M_n f(t) = (-1)^n \frac{1}{\phi_0(t)}\frac{d}{dt}\frac{1}{\phi_1(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{\phi_{n-1}(t)}\frac{d}{dt}\phi_0(t)\phi_1(t)\cdots\phi_{n-1}(t)f(t). \tag{9}$$

In formulas (8) and (9) the operation of differentiation applies to all that follows.‡

* For a simple proof of this fact see G. Pólya, loc. cit., p. 316.

† L. Schlesinger, *Lineare Differential-Gleichungen*, vol. 1, p. 58.

‡ Throughout this paper the independent variable for the operator L_n is x; for M_n, t. The expression $L_n f(t)$ means $L_n f(x)|_{x=t}$.

The functions $g_n(x, t)$ can be expressed in terms of the functions $\phi_i(x)$. For $g_n(x, t)$, considered as a function of x, satisfies the differential system

$$L_{n+1}u(x) = 0,$$

$$L_m u(t) = \begin{cases} 0, & m = 0, 1, \cdots, n-1, \\ 1, & m = n. \end{cases}$$

The system has a unique solution since the boundary conditions are equivalent to

$$u^{(m)}(t) = \begin{cases} 0, & m = 0, 1, \cdots, n-1, \\ 1, & m = n. \end{cases}$$

But by virtue of formula (8) the solution takes the form

$$\begin{aligned} u(x) = g_n(x, t) = {} & \frac{\phi_0(x)}{\phi_0(t) \cdots \phi_n(t)} \int_t^x \phi_1(x_1) \int_t^{x_1} \\ & \cdots \int_t^{x_{n-2}} \phi_{n-1}(x_{n-1}) \int_t^{x_{n-1}} \phi_n(x_n) dx_1 dx_2 \cdots dx_n, \end{aligned} \tag{10}$$

a formula which we shall also write as follows:

$$g_n(x, t) = \frac{\phi_0(x)}{\phi_0(t) \cdots \phi_n(t)} \int_t^x \phi_1(x) \int_t^x \phi_2(x) \int_t^x \cdots \int_t^x \phi_{n-1}(x) \int_t^x \phi_n(x) (dx)^n.$$

That this function satisfies the differential equation is obvious. That it satisfies the boundary conditions may be seen by forming the functions

$$L_m g_n(x, t) = \frac{\phi_0(x) \cdots \phi_m(x)}{\phi_0(t) \cdots \phi_n(t)} \int_t^x \phi_{m+1}(x) \int_t^x \cdots \int_t^x \phi_n(x) (dx)^{n-m}, \quad m < n,$$

$$L_n g_n(x, t) = \frac{\phi_0(x) \cdots \phi_n(x)}{\phi_0(t) \cdots \phi_n(t)},$$

and substituting $x = t$.

It is a familiar fact, and one that may be directly verified by use of formulas (6) and (9), that $g_n(x, t)$ considered as a function of t satisfies the adjoint differential system

$$M_{n+1} v(t) = 0, \tag{11}$$

$$v^{(m)}(x) = \begin{cases} 0, & m = 0, 1, 2, \cdots, n-1, \\ (-1)^n, & m = n. \end{cases} \tag{12}$$

But an argument similar to that given above shows that the solution of this system has the form

$$(13)\qquad \begin{aligned} v(t) &= g_n(x,t) \\ &= (-1)^n \frac{\phi_0(x)}{\phi_0(t)\cdots\phi_n(t)} \int_x^t \phi_n(t) \int_x^t \phi_{n-1}(t) \int_x^t \cdots \int_x^t \phi_1(t)(dt)^n. \end{aligned}$$

This formula has the advantage over (10) that it enables one to express $g_n(x, t)$ in terms of $g_{n-1}(x, t)$:

$$g_n(x,t) = \frac{1}{\phi_0(t)\cdots\phi_n(t)} \int_t^x \phi_0(t) \cdots \phi_n(t) g_{n-1}(x,t)dt.$$

By use of this formula the functions $g_n(x, t)$ may be computed step by step from the functions $\phi_i(x)$, the computations involving only one new integration for each new function $g_n(x, t)$.

It should be pointed out that for many purposes it is convenient to consider the functions $\phi_i(x)$ as the given functions instead of the $u_i(x)$. For if the $\phi_i(x)$ are given positive functions in (a, b), then a set of functions $u_i(x)$ possessing the property W in that interval is

$$u_i(x) = g_i(x,t) \qquad (i = 0,1,2, \cdots; a \leqq t \leqq b).$$

Evidently any function $\phi_i(x)$ may be multiplied by an arbitrary constant not zero without affecting the form of the series; for a glance at formulas (8) and (10) will show that neither the operators L_n nor the functions $g_n(x, t)$ will be thereby affected. For the special sequence (1′) we have

$$\begin{aligned} \phi_k(x) &= k \qquad (k = 1,2,3, \cdots), \\ \phi_0(x) &= 1. \end{aligned}$$

However, one is led equally well to Taylor's series by taking

$$\phi_k(x) = 1 \qquad (k = 0,1,2, \cdots).$$

5. **Remainder formulas.** Let us begin by deriving an exact remainder formula, the analogue of a well known formula for Taylor's series.* Set

$$R_n(x) = f(x) - L_0 f(t) g_0(x,t) - L_1 f(t) g_1(x,t) - \cdots - L_n f(t) g_n(x,t).$$

By Theorem I this function has a zero of order $(n+1)$ at least at $x=t$. Furthermore it satisfies the differential equation

$$L_{n+1} R_n(x) = L_{n+1} f(x).$$

* See for example E. Goursat, loc. cit., vol. 1, p. 209.

But it is known that the only solution of this equation vanishing with its first n derivatives at $x=t$ is

$$R_n(x) = \int_t^x g_n(x,t)L_{n+1}f(t)dt. \tag{14}$$

This gives the remainder formula desired:

$$\begin{aligned} f(x) = L_0f(x)g_0(x,t) + L_1f(t)g_1(x,t) + \cdots + L_nf(t)g_n(x,t) \\ + \int_t^x g_n(x,t)L_{n+1}f(t)dt. \end{aligned} \tag{15}$$

For the special sequence (1′) this becomes

$$f(x) = f(t) + f'(t)(x-t) + \cdots + f^{(n)}(t)\frac{(x-t)^n}{n!} + \int_t^x \frac{(x-t)^n}{n!}f^{(n+1)}(t)dt.$$

In the previous section we assumed that the functions $f(x)$ and $u_i(x)$ were of class C^∞. For the validity of the remainder formula (15) it is clearly sufficient to assume that $f(x)$, $u_0(x)$, $u_1(x)$, $\cdots$, $u_n(x)$ are of class C^{n+1} and that the Wronskians $W_0(x)$, $W_1(x)$, $\cdots$, $W_n(x)$ are positive in (a, b).

Let us now obtain remainder formulas analogous to certain other of the classical remainder formulas for Taylor's series. Let $F(s)$ be a function of class C' in the interval (a, b), and such that $F'(s)$ is not zero in the interval (t, x) except perhaps at the point t. Then formula (14) may evidently be written as

$$R_n(x) = \int_t^x \frac{g_n(x,s)}{F'(s)}L_{n+1}f(s)F'(s)ds.$$

We may now apply the first mean-value theorem for integrals,* and obtain

$$R_n(x) = \frac{g_n(x,\xi)}{F'(\xi)}L_{n+1}f(\xi)[F(x) - F(t)] \qquad (t < \xi < x,\ x < \xi < t). \tag{16}$$

This is the analogue of the remainder given by Schömilch,†

$$R_n(x) = \frac{f^{(n+1)}(\xi)(x-\xi)^n}{F'(\xi)n!}[F(x) - F(t)],$$

to which it reduces for the special sequence (1′).

* E. Goursat, loc. cit., vol. 1, p. 181. It is to be noted that $[g_n(x,s)\ L_{n+1}f(s)]/(F'(s))$ may be discontinuous at $s=t$. The ordinary treatments of the theorem do not admit this possibility, but it may be shown that the theorem is still applicable to this case; cf. G. D. Birkhoff, these Transactions, vol. 7 (1906), p. 115.

† For references see Encyklopädie der Mathematischen Wissenschaften, IIA2 (Pringsheim), § 11.

By specializing the function $F(s)$ a variety of remainders may be obtained. Let us take

$$F(s) = \int_t^s g_m(x,s)ds, \quad m \leqq n.$$

Then $F(s)$ obviously possesses the continuity properties imposed above. That it is a function of one sign in the open interval (t, x) may be seen by direct inspection of formula (10) or by the general theory of G. Pólya.* For, by formulas (11) and (12) we see that $g_m(x, s)$ considered as a function of s has a zero of order m at the point x and satisfies the differential equation

$$M_{m+1}v(s) = 0.$$

But no solution of this equation not identically zero can vanish more than m times in any interval in which the Property W holds. Consequently $g_m(x, s)$ is different from zero in (a, b) except at x. With this special choice of $F(s)$, (16) becomes

$$R_n(x) = \frac{g_n(x,\xi)}{g_m(x,\xi)} L_{n+1}f(\xi) \int_t^x g_m(x,s)ds. \tag{17}$$

This is the analogue of a remainder of Roche,†

$$R_n(x) = \frac{(x-\xi)^{n-m}(x-t)^{n+1}}{n!(m+1)} f^{(n+1)}(\xi),$$

to which it reduces for the sequence (1′).

By taking $m=n$, (17) becomes

$$R_n(x) = L_{n+1}f(\xi) \int_t^x g_n(x,s)ds,$$

and this is the analogue of the familiar Lagrange† remainder. Finally by taking $m=0$ we obtain

$$R_n(x) = \frac{g_n(x,\xi)}{g_0(x,\xi)} L_{n+1}f(\xi) \int_t^x g_0(x,s)ds,$$

as the analogue of Cauchy's† remainder,

$$R_n(x) = \frac{(x-\xi)^n(x-t)}{n!} f^{(n+1)}(\xi).$$

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* G. Pólya, loc. cit., p. 317.

† Encyklopädie, II A2, loc. cit.

A simpler remainder which also reduces to that of Cauchy for the sequence (1′) is

$$R_n(x) = g_n(x,\xi)L_{n+1}f(\xi)(x-t).$$

Let us sum up the results in

THEOREM II. *Let the functions $f(x)$, $u_0(x)$, $u_1(x)$, $\cdots$, $u_n(x)$ be of class C^{n+1}, and let the Wronskians $W_0(x)$, $W_1(x)$, $\cdots$, $W_n(x)$ be positive in the interval $a \leqq x \leqq b$. Then if t is a point of this interval,*

$$f(x) = L_0f(t)g_0(x,t) + L_1f(t)g_1(x,t) + \cdots + L_nf(t)g_n(x,t) + R_n(x), \tag{18}$$

where

$$g_k(x,t) = \left(\frac{1}{W_k(t)}\right)\begin{vmatrix} u_0(t) & u_1(t) & \cdots & u_k(t) \\ u_0'(t) & u_1'(t) & \cdots & u_k'(t) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ u_0^{(k-1)}(t) & u_1^{(k-1)}(t) & \cdots & u_k^{(k-1)}(t) \\ u_0(x) & u_1(x) & \cdots & u_k(x) \end{vmatrix},$$

$$L_kf(x) = \frac{W[u_0(x), \cdots, u_{k-1}(x), f(x)]}{W_{k-1}(x)}$$

$$(k = 1,2, \cdots, n+1\ ;\ L_0f(x) = f(x)),$$

and where $R_n(x)$ has one of the forms

$$R_n(x) = \int_t^x g_n(x,t)L_{n+1}f(t)dt,$$

$$R_n(x) = \frac{g_n(x,\xi)}{g_m(x,\xi)}L_{n+1}f(\xi)\int_t^x g_m(x,s)ds$$

$$(m \leqq n\ ;\ t < \xi < x\ ;\ x < \xi < t).$$

The function

$$N_m(x,t) = \int_t^x g_m(x,s)ds$$

that appears in the remainder may be expressed in a different form, which will be useful in what is to follow. From the form of the function it is seen to satisfy the following differential system when considered as a function of x:

$$L_{m+1}u(x) = 1,$$

$$u^{(k)}(t) = 0 \qquad (k = 0,1,2, \cdots, m).$$

But the unique solution of this system may also be written in the form

$$N_m(x,t) = \phi_0(x)\int_t^x \phi_1(x)\int_t^x \cdots \int_t^x \phi_m(x)\int_t^x \frac{(dx)^{m+1}}{\phi_0(x)\cdots\phi_m(x)}\,. \tag{19}$$

For the special sequence (1′) this is equal to $(x-t)^{m+1}/(m+1)!$.

6. **Generalized power series.** If in formula (18) n is allowed to become infinite, a series of the form

$$a_0g_0(x,t) + a_1g_1(x,t) + \cdots \tag{20}$$

results. Before discussing the behavior of the remainder as n becomes infinite, we discuss the general properties of a series of this type, a series which evidently reduces to a power series for the sequence (1′). In particular if $t=0$ is a point of (a, b), we shall set

$$g_n(x) = g_n(x,0) \qquad (n = 0,1,2,\cdots).$$

As has already been observed, no change is made in the series if any function $\phi_i(x)$ is multiplied by a non-vanishing constant. Consequently, no essential restriction will be introduced by the assumption, which will be made in the remainder of this paper, that $\phi_i(0)=1$. With this assumpton we may write

$$g_n(x) = \phi_0(x)\int_0^x \phi_1(x)\int_0^x \cdots \int_0^x \phi_n(x)(dx)^n. \tag{21}$$

In order that the series (20) may retain many of the formal properties of a power series we introduce

CONDITIONS A: (a) *The functions $\phi_i(x)$ are of class C^∞ in the interval $a\leqq x\leqq b$;*

(b) $$\phi_i(x) > 0 \qquad (i = 0,1,2,\cdots, a\leqq x\leqq b),$$

(c) $$\lim_{n=\infty}\frac{M_n}{m_n} = 1,$$

where

$$M_n = \text{maximum } \phi_n(x), \quad m_n = \text{minimum } \phi_n(x) \textit{ in } a\leqq x\leqq b.$$

In the case of the sequence (1′), $\phi_i(x)$ is constant, and the Conditions A are surely satisfied. It is a simple matter to construct other sequences of functions satisfying the conditions. For example, take

$$\phi_n(x) = e^{-x/n}.$$

Then

$$M_n = e^{-a/n}, \quad m_n = e^{-b/n},$$

and the conditions are evidently satisfied in any interval (a, b) however large.

We are now in a position to prove

THEOREM III. *If the functions $\phi_i(x)$ satisfy the Conditions* A *in* (a, b), *and if the series*

$$\sum_{n=0}^{\infty} c_n g_n(x,t), \qquad a \leqq t \leqq b, \tag{22}$$

converges for a value $x=x_0\neq t$ of that interval, then it converges absolutely in the interval $|x-t|<|x_0-t|$, $a\leqq x\leqq b$, and uniformly in any closed interval included therein. If the sum of the series is denoted by $f(x)$, then

$$L_k f(x) = \sum_{n=0}^{\infty} c_n L_k g_n(x,t) \tag{23}$$

$$(k = 0,1,2,\cdots;\ |x-t| < |x_0 - t|\ ;\ a \leqq x \leqq b).$$

Since the series (22) converges for $x=x_0$, it follows that there exists a constant M independent of n for which

$$|c_n g_n(x_0,t)| < M.$$

We are thus led immediately to a dominant series for (22),

$$\sum_{n=0}^{\infty} c_n g_n(x,t) \ll M \sum_{n=0}^{\infty} \frac{|g_n(x,t)|}{|g_n(x_0,t)|}.$$

We now obtain a more convenient form for $g_n(x, t)$ by successive applications of the mean-value theorem for integrals:

$$g_n(x,t) = \frac{\phi_0(x)\phi_1(\xi_1)\phi_2(\xi_2)\cdots\phi_n(\xi_n)}{\phi_0(t)\phi_1(t)\phi_2(t)\cdots\phi_n(t)}\,\frac{(x-t)^n}{n!},$$

$$t < \xi_n < \xi_{n-1} < \cdots < \xi_1 < x,$$

$$t > \xi_n > \xi_{n-1} > \cdots > \xi_1 > x.$$

Here the first line of inequalities holds if $t<x$; the second if $t>x$. Now making use of the upper and lower bounds M_n and m_n of ϕ_n in (a, b), we see that

$$|g_n(x,t)| < \frac{M_0M_1\cdots M_n}{m_0m_1\cdots m_n}\,\frac{|x-t|^n}{n!},$$

$$|g_n(x_0,t)| > \frac{m_0m_1\cdots m_n}{M_0M_1\cdots M_n}\,\frac{|x_0-t|^n}{n!},$$

$$\sum_{n=0}^{\infty} c_n g_n(x,t) \ll M \sum_{n=0}^{\infty}\left(\frac{M_0M_1\cdots M_n}{m_0m_1\cdots m_n}\right)^2 \frac{|x-t|^n}{|x_0-t|^n}.$$

The test ratio

$$\left(\frac{M_{n+1}}{m_{n+1}}\right)^2 \left|\frac{x-t}{x_0-t}\right|$$

of the dominant series has the limit $|x-t|/|x_0-t|$ as n becomes infinite by Condition A (c). The first part of the theorem is thus established. It remains to show that the operation term by term by L_k is permissible. Now

$$(24)\quad \begin{aligned} L_k g_n(x,t) &= \frac{\phi_0(x)\cdots\phi_k(x)}{\phi_0(t)\cdots\phi_n(t)}\int_t^x \phi_{k+1}(x)\int_t^x\cdots\int_t^x \phi_n(x)(dx)^{n-k}, \quad n \geqq k, \\ L_k g_n(x,t) &= \frac{\phi_0(x)\cdots\phi_k(x)\phi_{k+1}(\xi_{k+1})\cdots\phi_n(\xi_n)}{\phi_0(t)\cdots\phi_n(t)}\frac{(x-t)^{n-k}}{(n-k)!}, \end{aligned}$$

$$t < \xi_n < \xi_{n-1} < \cdots < \xi_{k+1} < x,$$

$$t > \xi_n > \xi_{n-1} > \cdots > \xi_{k+1} > x.$$

Hence

$$\sum_{n=k}^{\infty} c_n L_k g_n(x,t) \ll M \sum_{n=k}^{\infty}\frac{|L_k g_n(x,t)|}{|g_n(x_0,t)|}$$

$$\ll M\sum_{n=k}^{\infty}\left(\frac{M_0\cdots M_n}{m_0\cdots m_n}\right)^2\frac{|x-t|^{n-k}}{|x_0-t|^n}\frac{n!}{(n-k)!}.$$

Consequently the series (23) is uniformly convergent for $|x-t|\leqq r$, $a\leqq x\leqq b$, where $r<|x_0|$. This is sufficient to establish the result stated.

As a result of this theorem it follows that there exists an interval of convergence for the series extending equal distances on either side of t (provided the length of the interval of definition (a, b) permits). In particular, the interval may reduce to a single point, or it may be the entire interval (a, b) (which in turn may, in special cases, be the entire x-axis). The following examples will show that all of these cases are possible. Take $\phi_n(x)=e^{-x/n}$. Then

$$\sum_{n=0}^{\infty}(n!)^2 g_n(x) \textit{ diverges except for } x = 0;$$

$$\sum_{n=0}^{\infty} n!g_n(x) \quad \textit{converges for } |x| < 1,$$

$$\textit{diverges for } |x| > 1;$$

$$\sum_{n=0}^{\infty} g_n(x) \quad \textit{converges for all } x.$$

Theorem III has a further important consequence. If in equations (23) we set $x=t$, we see that

$$c_k = L_k f(t).$$

Since the coefficients c_k are uniquely determined by the values of $f(x)$ and its derivatives at $x=t$, it follows that the development of a function $f(x)$ in a series (22) is unique.

7. **A generalization of Abel's theorem.** If a series (22) has an interval of convergence $(-r, r)$,* then by Theorem III it has a continuous sum in the interval $-r<x<r$. As in the case of power series the series may or may not converge at the extremities of the interval. We shall show that if (22) converges at r(or $-r$), then the sum of the series is continuous in the interval $-r<x\leqq r$ (or $-r\leqq x<r$) by use of the following

LEMMA. *If the functions* $\phi_n(x)$ *satisfy the conditions* A (a), (b), *then the determinant*

$$\Delta = \begin{vmatrix} g_{n-1}(x) & g_n(x) \\ g_{n-1}(y) & g_n(y) \end{vmatrix}$$

is positive or negative according as $0<x<y$ *or* $0>x>y$.

First it will be shown that $\Delta\neq 0$. If Δ were equal to zero for two values x_0 and y_0 distinct from each other and from the origin, it would be possible to determine constants c_1 and c_2 not both zero such that the function

$$\phi(x) = c_1 g_{n-1}(x) + c_2 g_n(x) \tag{25}$$

would vanish at x_0 and y_0. But $g_{n-1}(x)$ and $g_n(x)$ both vanish $(n-1)$ times at the origin so that $\phi(x)$ would have at least $(n+1)$ zeros in $(-a, a)$. This however is impossible. For, according to the general results of Pólya already cited, no linear combination of $g_0(x), g_1(x), \cdots, g_n(x)$ not identically zero can vanish $(n+1)$ times in an interval in which the Property W holds. Hence $\Delta\neq 0$.

It remains to discuss the sign of Δ. Regard y as fixed, so that Δ becomes a function of x alone. Evidently

$$\lim_{x\to y} \frac{\Delta}{y-x} = W(y) = \begin{vmatrix} g_{n-1}(y) & g_n(y) \\ g'_{n-1}(y) & g'_n(y) \end{vmatrix}.$$

We shall show presently that $W(y)>0$ for all values of y different from zero in $(-a, a)$. This will be sufficient to establish the Lemma.

* Throughout this section we assume that $a<-r<r<b$; t is taken equal to zero for simplicity.

For, if x is allowed to approach a positive value of y through values less than y, then $\Delta/(y-x)$ remains a function of one sign (with the same sign as Δ), and approaches a positive value. The variable Δ must therefore have been positive. By allowing x to approach a negative y through values between y and zero, we see that

$$\Delta < 0, \quad y < x < 0.$$

To prove that $W(y)>0$ throughout $(-a, a)$ except at the origin, first note that

$$W^{(k)}(0) = 0 \qquad (k = 0,1, \cdots, 2n-3),$$

$$W^{(2n-2)}(0) = \begin{vmatrix} g_{n-1}^{(n-1)}(0) & g_n^{(n-1)}(0) \\ g_{n-1}^{(n)}(0) & g_n^{(n)}(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ g_{n-1}^{(n)}(0) & 1 \end{vmatrix} = 1$$

by virtue of relations (6). Hence

$$W(y) = \frac{y^{2n-2}}{(2n-2)!} + \frac{y^{2n-1}}{(2n-1)!}W^{(2n-1)}(\xi), \qquad 0 < \xi < y, \ 0 > \xi > y.$$

This shows that $W(y)>0$ for values of y sufficiently near the origin. But the same argument used above to show that Δ is different from zero may be used to show that $W(y)$ is different from zero away from the origin. The Lemma is thus completely established.

By use of this Lemma it is possible to prove

THEOREM IV. *Let the function $\phi_i(x)$ satisfy Conditions* A *in $(-a, a)$, and let the interval of convergence of the series*

$$\sum_{n=0}^{\infty} c_n g_n(x)$$

be $(-r, r)$. Then if the series converges for $x=r$ (or $x=-r$), its sum is continuous in the interval $-r<x\leqq r$ (or $-r\leqq x<r$).

Since the series converges for $x=r$, then to an arbitrary positive ϵ there corresponds a number m such that

$$\left|c_{m+1}g_{m+1}(r) + \cdots + c_{m+p}g_{m+p}(r)\right| < \epsilon \qquad (p = 1,2,3, \cdots).$$

Now by the Lemma the set of values

$$\frac{g_0(x)}{g_0(r)}, \quad \frac{g_1(x)}{g_1(r)}, \quad \frac{g_2(x)}{g_2(r)}, \cdots, \quad 0 < x < r,$$

forms a decreasing set. Hence by Abel's lemma*

* E. Goursat, *Cours d'Analyse Mathématique*, vol. 1, p. 182.

$$\left| c_{m+1}g_{m+1}(r)\,\frac{g_{m+1}(x)}{g_{m+1}(r)} + \cdots + c_{m+p}g_{m+p}(r)\,\frac{g_{m+p}(x)}{g_{m+p}(r)} \right| < \epsilon\frac{g_{m+1}(x)}{g_{m+1}(r)},$$

$$\leqq \epsilon\frac{g_0(x)}{g_0(r)} \leqq \epsilon M, \quad 0 \leqq x \leqq r,$$

where M is the maximum of $g_0(x)/g_0(r)$ in $0\leqq x\leqq r$. Consequently the series converges uniformly in $0\leqq x\leqq r$, and represents a continuous function there. That the sum is continuous in $-r<x\leqq 0$ follows from Theorem III. A similar proof shows that if the series converges at $-r$, then the sum is continuous in $-r\leqq x<r$.

8. **Generalization of the process of analytic continuation.** Let us first obtain formulas analogous to the binomial formulas

$$(x-t)^n = x^n - nx^{n-1}t + \frac{n(n-1)}{2!}x^{n-2}t^2 - \cdots + (-1)^n t^n, \tag{26}$$

$$x^n = t^n + nt^{n-1}(x-t) + \frac{n(n-1)}{2!}t^{n-2}(x-t)^2 + \cdots + (x-t)^n. \tag{27}$$

Let t and u be two distinct points of the interval (a, b). Since $g_n(x, t)$ and $g_n(x, u)$ are both linear combinations of $u_0(x), u_1(x), \cdots, u_n(x)$,

$$g_n(x,u) = c_0g_0(x,t) + c_1g_1(x,t) + \cdots + c_ng_n(x,t).$$

The constants c_i may be determined by use of formulas (7). The result is

$$g_n(x,u) = L_0g_n(t,u)g_0(x,t) + L_1g_n(t,u)g_1(x,t) + \cdots + L_ng_n(t,u)g_n(x,t). \tag{28}$$

For the particular sequence (1′) this reduces to (26) with $t=0$, $u=t$, and to (27) with $u=0$, $t=t$.

In order to generalize the process of analytic continuation we begin with a consideration of the double series

$$\begin{aligned} &c_0L_0g_0(t,u)g_0(x,t) \\ &+ c_1L_0g_1(t,u)g_0(x,t) + c_1L_1g_1(t,u)g_1(x,t) \\ &+ c_2L_0g_2(t,u)g_0(x,t) + c_2L_1g_2(t,u)g_1(x,t) + c_2L_2g_2(t,u)g_2(x,t) \\ &+ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &+ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ . \end{aligned} \tag{29}$$

Let us suppose that the interval of convergence of the series

$$\sum_{n=0}^{\infty} c_ng_n(x,u) \tag{30}$$

is $|x-u|<r, a\leqq x\leqq b$. It will now be possible to show the double series (29) absolutely convergent in a certain interval. The general term of that series is

$$c_nL_kg_n(t,u)g_k(x,t), \quad 0 \leqq k \leqq n.$$

Assuming Conditions A, we may obtain an upper bound for this term as follows:

$$|g_k(x,t)| \leqq \frac{M_0M_1\cdots M_k}{m_0m_1\cdots m_k}\,\frac{|x-t|^k}{k!},$$

$$|L_kg_n(t,u)| \leqq \frac{M_0M_1\cdots M_n}{m_0m_1\cdots m_n}\,\frac{|t-u|^{n-k}}{(n-k)!}, \qquad n \geqq k.$$

Let x_0 be a point in the interval of convergence of the series (30). Then there exists a constant M independent of n for which

$$|c_ng_n(x_0,u)| < M.$$

Hence we have

$$|g_n(x_0,u)| > \frac{m_0m_1\cdots m_n}{M_0M_1\cdots M_n}\,\frac{|x_0-u|^n}{n!},$$

$$|c_n| < \frac{MM_0\cdots M_n}{m_0\cdots m_n}\,\frac{n!}{|x_0-u|^n}.$$

Consequently, observing that $M_k/m_k\geqq 1$,

$$|c_nL_kg_n(t,u)g_k(x,t)| < M\left(\frac{M_0M_1\cdots M_n}{m_0m_1\cdots m_n}\right)^3\frac{n!}{k!(n-k)!}\,\frac{|x-t|^k|t-u|^{n-k}}{|x_0-u|^n}.$$

We are thus led to a dominant double series, which will now be shown convergent under certain conditions. First form the sum of the nth row of this series:

$$M\left(\frac{M_0\cdots M_n}{m_0\cdots m_n}\right)^3\frac{1}{|x_0-u|^n}\sum_{k=0}^{n}\frac{n!|x-t|^k|t-u|^{n-k}}{k!(n-k)!}$$

$$= M\left(\frac{M_0\cdots M_n}{m_0\cdots m_n}\right)^3\frac{1}{|x_0-u|^n}(|t-u|+|x-t|)^n.$$

Then form the sum of the row values

$$M\sum_{n=0}^{\infty}\left(\frac{M_0M_1\cdots M_n}{m_0m_1\cdots m_n}\right)^3\left(\frac{|t-u|+|x-t|}{|x_0-u|}\right)^n.$$

The test ratio of this series is

$$\left(\frac{M_n}{m_n}\right)^3 \frac{|t-u|+|x-t|}{|x_0-u|},$$

and by Condition A (c) this has the limit

$$\frac{|t-u|+|x-t|}{|x_0-u|}$$

as n becomes infinite. Consequently the series (29) is absolutely convergent if

$$|t-u|+|x-t|<|x_0-u|, \quad a \leqq x \leqq b.$$

The sum of the series may be obtained by summing by rows or by columns. In the one case, using formula (28), we find the sum to be the convergent series

$$\sum_{n=0}^{\infty} c_n g_n(x,u). \tag{31}$$

In the other case, the sum is found to be

$$\sum_{n=0}^{\infty} L_n f(t) g_n(x,t), \tag{32}$$

where $f(x)$ is defined as the sum of the series (31). That

$$L_k f(t) = \sum_{n=0}^{\infty} c_n L_k g_n(t,u)$$

follows from Theorem III. We thus have two representations for $f(x)$, the first of which, (31), holds in $|x-u|<r$, $a \leqq x \leqq b$, and the second of which, (32), holds in $|x-t|<r-|t-u|$, $a \leqq x \leqq b$. Conceivably, series (32) may converge in a larger interval, in which case an extension or prolongation of $f(x)$ would be at hand. We sum up the results in

THEOREM V. *If the functions $\phi_i(x)$ satisfy Conditions* A *in* (a, b), *and if*

$$f(x) = \sum_{n=0}^{\infty} c_n g_n(x,u), \qquad |x-u|<r, \quad a \leqq x \leqq b, \ a \leqq u \leqq b,$$

then

$$f(x) = \sum_{n=0}^{\infty} L_n f(t) g_n(x,t)$$

for all x and t satisfying the relation

$$|x-t|<r-|t-u|, \quad a \leqq x \leqq b, \quad a \leqq t \leqq b.$$

9. **A generalization of a theorem of S. Bernstein.** We shall now obtain a necessary and sufficient condition for the representation of a function $f(x)$ in a series of the type in question. The method consists in generalizing a familiar theorem of S. Bernstein.* The results to be proved are stated in

THEOREM VI. *Let the functions ϕ_i satisfy Conditions* A *in (a, b). Then a necessary and sufficient condition that a function $f(x)$, defined in the interval $a \leqq x < b$, can be represented by a series*

$$f(x) = \sum_{n=0}^{\infty} L_n f(a) g_n(x,a), \quad a \leqq x < b, \tag{33}$$

is that $f(x)$ be the difference of two functions of class C^∞ in $a \leqq x < b$,

$$f(x) = \phi(x) - \psi(x),$$

such that

$$L_n\phi(x) > 0 \text{ or } \phi(x) \equiv 0\,; \quad L_n\psi(x) > 0, \text{ or } \psi(x) \equiv 0, \quad a < x < b$$
$$(n = 0, 1, 2, \cdots).$$

We begin by proving the necessity of the condition. We suppose that

$$f(x) = \sum_{n=0}^{\infty} L_n f(a) g_n(x,a), \quad a \leqq x < b.$$

By Theorem III this series is absolutely convergent in $a \leqq x < b$, and hence we may set

$$\phi(x) = \sum_{n=0}^{\infty} |L_n f(a)| g_n(x,a),$$

$$\psi(x) = \sum_{n=0}^{\infty} \{ |L_n f(a)| - L_n f(a) \} g_n(x,a),$$

$$f(x) = \phi(x) - \psi(x), \qquad a \leqq x < b.$$

Again using the results of Theorem III, we have

$$L_k\phi(x) = \sum_{n=0}^{\infty} |L_n f(a)| L_k g_n(x,a) \qquad (k = 0, 1, 2, \cdots),$$

$$L_k\psi(x) = \sum_{n=0}^{\infty} \{ |L_n f(a)| - L_n f(a) \} L_k g_n(x,a).$$

* S. Bernstein, *Sur la définition et les propriétés des fonctions analytiques d'une variable réelle*, Mathematische Annalen, vol. 75 (1914), p. 449.

By reference to (24) it is seen that every non-vanishing term of each of these series is positive throughout the interval $a<x<b$. The necessity of the condition is thus established.

Conversely, suppose that $f(x)=\phi(x)-\psi(x)$, where $\phi(x)$ and $\psi(x)$ satisfy the conditions of the theorem. It will be enough to show that $\phi(x)$ can be represented in a series (33), for a similar proof will apply to $\psi(x)$; and, since the operators L_n are linear, we will then obtain a representation of the form desired for $f(x)$ by subtracting the series for $\phi(x)$ and $\psi(x)$.

We suppose that $\phi(x)$ is not identically zero, for otherwise the result is obvious. Choose a point x_0 of the interval $a<x<b$, and consider the following exact remainder formula:

$$\phi(x_0) = L_0\phi(t)g_0(x_0,t) + L_1\phi(t)g_1(x_0,t) + \cdots + L_n\phi(t)g_n(x_0,t)$$

$$+ \int_t^{x_0} g_n(x_0,t)L_{n+1}\phi(t)dt,$$

where $a<t<x_0$. Since the functions $\phi_n(x)$ are all positive, the functions $g_n(x_0, t)$ are all positive. By hypothesis $L_{n+1}\phi(t)$ is positive. Consequently the above integral is surely positive, as is each term on the right-hand side of the equation. Hence

$$\phi(x_0) > L_n\phi(t)g_n(x_0,t),$$

$$L_n\phi(t) < \frac{\phi(x_0)}{g_n(x_0,t)} < \phi(x_0)\frac{M_0M_1\cdots M_n}{m_0m_1\cdots m_n}\frac{n!}{(x_0-t)^n}. \tag{34}$$

Now referring to Theorem II and to formula (19), we see that

$$\phi(x) = L_0\phi(t)g_0(x,t) + L_1\phi(t)g_1(x,t) + \cdots + L_n\phi(t)g_n(x,t) + R_n,$$

$$R_n = L_{n+1}\phi(\xi)\phi_0(x)\int_t^x\phi_1(x)\int_t^x\phi_2(x)\int_t^x\cdots\int_t^x\phi_n(x)\int_t^x\frac{(dx)^{n+1}}{\phi_0(x)\cdots\phi_n(x)},$$

$$t<\xi<x,$$

$$t>\xi>x,$$

$$R_n = L_{n+1}\phi(\xi)\frac{\phi_0(x)\phi_1(\xi_1)\cdots\phi_n(\xi_n)}{\phi_0(\xi_{n+1})\phi_1(\xi_{n+1})\cdots\phi_n(\xi_{n+1})}\frac{(x-t)^{n+1}}{(n+1)!},$$

$$t<\xi_{n+1}<\xi_n<\cdots<\xi_1<x,$$

$$t>\xi_{n+1}>\xi_n>\cdots>\xi_1>x.$$

Setting $t=\xi$ in (34), we have

$$|R_n| < \phi(x_0)\left(\frac{M_0\cdots M_{n+1}}{m_0\cdots m_{n+1}}\right)^2\frac{|x-t|^{n+1}}{(x_0-\xi)^{n+1}}.$$

Evidently the remainder approaches zero as n becomes infinite if

$$|x-t| < \frac{x_0 - t}{2}, \qquad a \leqq x.$$

If now t is allowed to approach a, the following expansion results:

$$\phi(x) = \sum_{n=0}^{\infty} L_n\phi(a)g_n(x,a), \qquad 0 \leqq x - a < \frac{x_0 - a}{2}\,. \tag{35}$$

But the series (35) converges in a larger interval. For

$$\sum_{n=0}^{\infty} L_n\phi(a)g_n(x,a) \ll \phi(x_0)\sum_{n=0}^{\infty}\left(\frac{M_0\cdots M_n}{m_0\cdots m_n}\right)^2\frac{|x-a|^n}{(x_0-a)^n}, \tag{36}$$

and the dominant series converges for $|x-a|<x_0-a$. It remains only to show that the sum of the series is $\phi(x)$ throughout the interval $a\leqq x<b$.

Denote the sum of the series (36) by $H(x)$. Then $H(x)=\phi(x)$ for $a\leqq x<(a+x_0)/2$. Choose a point t in this interval near to $(x_0+a)/2$. We have seen above that

$$\phi(x) = \sum_{n=0}^{\infty} L_nH(t)g_n(x,t) = \sum_{n=0}^{\infty} L_n\phi(t)g_n(x,t), \quad |x-t| < \frac{x_0-t}{2}, \quad x \geqq a.$$

But by Theorem V

$$H(x) = \sum_{n=0}^{\infty} L_nH(t)g_n(x,t), \quad |x-t| < x_0 - t, \quad x \geqq a.$$

Consequently $H(x)=\phi(x)$ for $a\leqq x<(x_0+t)/2$. Now choose a point t' in this interval near to $(x_0+t)/2$, and proceed as before to show that $H(x)=\phi(x)$ in $a\leqq x<(x_0+t')/2$. By continuing the process we see that $H(x)$ and $\phi(x)$ coincide in the entire interval $a\leqq x<x_0$. But x_0 was an arbitrary point of $a<x<b$. Consequently equation (33) holds in this interval, and the proof is complete.

10. **The expansion of an arbitrary analytic function.** After imposing further conditions on the functions $\phi_i(x)$ it will be found possible to represent an arbitrary analytic function in a generalized power series. We define

CONDITIONS B. (a) *Conditions* A *are satisfied in* (a, b);

(b) $$\frac{d^k}{dx^k}\left(\frac{1}{\phi_n(x)}\right) \geqq 0 \qquad (k = 1,2,3,\cdots;\ n = 0,1,2,3,\cdots;\ a \leqq x \leqq b).$$

We now state a very simple lemma, the proof of which follows immediately from Leibniz's rule for the differentiation of a product.

Lemma. *If $f(x)$ is positive with positive derivatives of all orders, and if $\phi(x)$ is positive with derivatives of all orders that are positive or zero, then $(d/dx) \cdot (f(x)\cdot\phi(x))$ is positive with all its derivatives.*

We are now in a position to prove

Theorem VII. *If Conditions* B *are satisfied in (a, b), and if $f(x)$ is analytic in $a<x<b$, then*

$$f(x) = \sum_{n=0}^{\infty} L_n f(t) g_n(x,t), \qquad a < t < b,$$

the series being convergent in some neighborhood of t.

Since $f(x)$ is analytic at t, it can be represented as a power series

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(t)\frac{(x-t)^n}{n!}, \qquad |x-t| < r.$$

Then it follows that the expansion

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}\left(t-\frac{r}{3}\right)\left(x-t+\frac{r}{3}\right)^n$$

is certainly valid in the interval $t-r<x<t+r/3$.

Now set

$$g(x) = \sum_{n=0}^{\infty} \left|f^{(n)}\left(t-\frac{r}{3}\right)\right|\left(x-t+\frac{r}{3}\right)^n,$$

$$h(x) = \sum_{n=0}^{\infty} \left\{\left|f^{(n)}\left(t-\frac{r}{3}\right)\right| - f^{(n)}\left(t-\frac{r}{3}\right)\right\}\left(x-t+\frac{r}{3}\right)^n,$$

so that

$$f(x) = g(x) - h(x),$$

$g(x)$ and $h(x)$ being functions that are either identically zero or positive with all their derivatives in $t-r/3<x<t+r/3$. The trivial case in which $g(x)$ or $h(x)$ is identically zero may be discarded. Now by making successive applications of the Lemma it is seen that

$$L_n g(x) > 0,\ L_n h(x) > 0 \qquad \left(n = 0,1,2,\cdots;\ t-\frac{r}{3} < x < t+\frac{r}{3}\right).$$

Consequently Theorem VI may be applied to give

$$g(x) = \sum_{n=0}^{\infty} L_n g\left(t - \frac{r}{3}\right) g_n\left(x, t - \frac{r}{3}\right), \quad t - \frac{r}{3} \leqq x < t + \frac{r}{3},$$

$$h(x) = \sum_{n=0}^{\infty} L_n h\left(t - \frac{r}{3}\right) g_n\left(x, t - \frac{r}{3}\right).$$

Finally, we make use of Theorem V, and see that

$$g(x) = \sum_{n=0}^{\infty} L_n g(t) g_n(x,t),$$

$$h(x) = \sum_{n=0}^{\infty} L_n h(t) g_n(x,t),$$

$$f(x) = \sum_{n=0}^{\infty} L_n f(t) g_n(x,t), \qquad |x - t| < r.$$

The theorem is thus established.

It should be pointed out that Conditions B are not so strong as to exclude the case of Taylor's development. For, they are surely satisfied for $\phi_n(x) = 1$. Moreover, other sets of functions $\phi_n(x)$ exist satisfying the conditions. Witness the set

$$\phi_n(x) = e^{-x/n}.$$

11. **Teixeira's series.** In the introduction reference was made to certain series studied by Teixeira. We wish to show by a consideration of the sequence (1'') how these series arise naturally as a generalization of Taylor's series. In order that the Wronskians $W_n(x)$ may all be positive we change the sign of certain of the functions of the sequence (1''), an alteration that will not affect the form of the series. Consider then the sequence

$$(37) \qquad 1, \sin x, -\cos x, -\sin 2x, \cos 2x, \cdots, (-1)^{n-1}\sin nx, \\ (-1)^n \cos nx, \cdots.$$

The operators L_{2n+1} corresponding to this sequence have a particularly simple form:

$$L_{2n+1} = D(D^2 + 1^2)(D^2 + 2^2) \cdots (D^2 + n^2) \qquad (n = 0, 1, 2, \cdots),$$

where D indicates the operation of differentiation. The operators L_{2n} are more complicated. Direct computations show that

$$W_{2n} = n![(2n-1)!]^2[(2n-3)!]^2 \cdots [3!]^2.$$

By definition of the operator L_{2n+1} we have

$$L_{2n+1}(-1)^n \sin (n+1)x$$
$$= \frac{W(1, \sin x, \cos x, \cdots, \sin nx, \cos nx, (-1)^n \sin (n+1)x)}{W_{2n}} = \frac{W_{2n+1}}{W_{2n}},$$

whence

$$\begin{aligned} W_{2n+1} &= W_{2n} D(D^2+1^2) \cdots (D^2+n^2)(-1)^n \sin (n+1)x \\ &= W_{2n}(2n+1)! \cos (n+1)x. \end{aligned}$$

Hence the Wronskians $W_n(x)$ are all positive at the origin, and the functions of approximation, $g_n(x)$, all exist. We shall show that

$$g_{2n}(x) = \frac{2^n}{(2n)!}[1 - \cos x]^n, \quad g_{2n+1}(x) = \frac{2^n}{(2n+1)!}[1 - \cos x]^n \sin x.$$

By a familiar formula of trigonometry we have

$$\frac{2^n}{(2n)!}[1 - \cos x]^n = \sum_{k=-n}^{n} \frac{(-1)^k \cos kx}{(n-k)!(n+k)!}.$$

This function clearly satisfies the differential equation

$$L_{2n+1} u(x) = 0. \tag{38}$$

Moreover, it satisfies the boundary conditions

$$u^{(k)}(0) = 0 \quad (k = 0, 1, 2, \cdots, 2n-1); \quad u^{(2n)}(0) = 1. \tag{39}$$

But the differential system (38) (39) has only one solution, the function of approximation $g_{2n}(x)$.

By noting that

$$\begin{aligned} \frac{2^n}{(2n+1)!}[1 - \cos x]^n \sin x &= \frac{d}{dx} \frac{2^{n+1}}{(2n+2)!}[1 - \cos x]^{n+1} \\ &= \sum_{k=-n-1}^{n+1} \frac{(-1)^{k+1} k \sin kx}{(n+1-k)!(n+1+k)!} \end{aligned}$$

it is seen that this function satisfies the system

$$L_{2n+2} v(x) = 0,$$
$$v^{(k)}(0) = 0 \quad (k = 0, 1, 2, \cdots, 2n); \quad v^{(2n+1)}(0) = 1,$$

and consequently must be $g_{2n+1}(x)$.

In the expansion of the function $f(x)$, the coefficients of the terms $g_{2n}(x)$ will involve the complicated differential operator L_{2n}. We may, however,

express this coefficient in terms of a simpler operator of order $2n$. In doing this use will be made of the functions $\phi_n(x)$ which will now be computed:

$$\phi_0(x) = 1, \quad \phi_1(x) = \cos x,$$

$$\phi_{2n}(x) = \frac{n}{\cos^2 nx}, \quad \phi_{2n+1}(x) = 2(2n+1)\cos(n+1)x\cos nx$$
$$(n = 1,2,3,\cdots).$$

Evidently,

$$L_{2n+2}f(x) = \phi_0(x)\cdots\phi_{2n+1}(x)\frac{d}{dx}\frac{L_{2n+1}f(x)}{\phi_0(x)\cdots\phi_{2n+1}(x)}$$
$$= \frac{(\cos(n+1)x)DL_{2n+1}f(x) + (n+1)(\sin(n+1)x)L_{2n+1}f(x)}{\cos(n+1)x}.$$

Consequently it follows that,

$$L_{2n+2}f(0) = D^2(D^2+1^2)(D^2+2^2)\cdots(D^2+n^2)f(0).$$

The expansion of $f(x)$ now takes the form

$$f(x) \sim \sum_{n=0}^{\infty} A_n\frac{2^n}{(2n)!}[1-\cos x]^n + B_n\frac{2^n}{(2n+1)!}[1-\cos x]^n \sin x,$$
$$A_n = D^2(D^2+1^2)(D^2+2^2)\cdots(D^2+(n-1)^2)f(0),$$
$$B_n = D(D^2+1^2)(D^2+2^2)\cdots(D^2+n^2)f(0).$$

Although the sequence (37) does not satisfy the Conditions A directly, a simple substitution reduces the series (40) to one for which these conditions are satisfied. Indeed we shall see that the substitution $y=\sin(x/2)$ reduces the series to the sum of two Taylor's series, so that the convergence can be easily discussed.

An alternative form of the series is obviously

$$f(x) \sim \sum_{n=0}^{\infty} A_n \frac{2^{2n}}{(2n)!}\left(\sin\left(\frac{x}{2}\right)\right)^{2n}$$
$$+ B_n\frac{2^{2n+1}}{(2n+1)!}\left(\sin\left(\frac{x}{2}\right)\right)^{2n+1}\cos\left(\frac{x}{2}\right).$$

If the change of variable $x/2=y$ is made, the form of the series employed by Teixeira* is obtained.

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* For reference see § 1.

Now any function $f(x)$ analytic in the neighborhood of $x=0$ can be expanded in a series of this type for a sufficiently small neighborhood of $x=0$. For, if

$$\phi(x) = \frac{f(x) + f(-x)}{2}, \quad \psi(x) = \frac{f(x) - f(-x)}{2}, \quad \phi(x) + \psi(x) = f(x),$$

then the functions

$$\phi(2 \sin^{-1} y) \quad \text{and} \quad \frac{\psi(2 \sin^{-1} y)}{\cos \sin^{-1} y}$$

are both analytic in some neighborhood $|y|<\delta$ of $y=0$. Hence they can be expanded in powers of y:

$$\phi(2 \sin^{-1} y) = \sum_{n=0}^{\infty} a_{2n} y^{2n}, \quad |y| < \delta,$$

$$\frac{\psi(2 \sin^{-1} y)}{\cos \sin^{-1} y} = \sum_{n=0}^{\infty} b_{2n+1} y^{2n+1}, \quad |y| < \delta.$$

We have now only to make the substitution $y=\sin(x/2)$ in these series and to add in order to be assured that $f(x)$ can be expanded in a series (40) in some neighborhood of the origin. For simplicity expansion have been considered in the neighborhood of the origin, but the results clearly hold for an arbitrary point.

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A PROBLEM IN THE CALCULUS OF VARIATIONS WITH AN INFINITE NUMBER OF AUXILIARY CONDITIONS*

BY

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Introduction

The significance of the calculus of variations as a focal point of analysis has been emphasized by Hilbert and his school, and its intimate connection with the theories of mechanics, differential equations, integral equations, and quadratic forms in an infinite number of variables, has been used to the mutual benefit of all these disciplines. From one standpoint the problems of the calculus of variations may be regarded as problems of ordinary maxima and minima in a denumerable or non-denumerable infinity of independent variables; the imposition of a finite number of auxiliary conditions would then be equivalent to reducing the infinity of variables by a finite number. It is natural to inquire what will happen when a denumerable *infinity* of auxiliary conditions are imposed on the function involved in the integral to be minimized. In various branches of mathematics much light has been thrown on problems by a generalization from the finite to the infinite and it may reasonably be expected that there will be additional insight into the problems of the calculus of variations by the development of a similar extension.

This paper undertakes to make a beginning of such a study by treating a particular problem which has for its Euler condition a differential equation central in mathematical physics. Some of the results will appear as natural generalizations of criteria already known, while others seem in contradiction to them.

The problem to be studied is intimately related to one discussed earlier† by the author in which a *finite* number of auxiliary conditions were imposed. That discussion concerned the solutions of the equation

$$(0.1) \qquad L(u) \equiv (p(x)u'(x))' + q(x)u(x) + \lambda k(x)u(x) = 0, \quad p > 0,$$

subject to the boundary conditions

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† *Das Jacobische Kriterium der Variationsrechnung und die Oszillationseigenschaften linearer Differentialgleichungen 2. Ordnung*, Mathematische Annalen, vol. 68, p. 279.

$$u(0) = u(1) = 0. \tag{0.2}$$

There are three distinct cases of the equation (0.1) which may be distinguished as follows:

(i). *Orthogonal case.* When $k(x)$ is of one sign; for example, $k(x)$ is positive or zero and equal to zero only at a finite number of points in the interval. The system (0.1), (0.2) has an infinite number of normalized characteristic solutions $U_1, U_2, \cdots$, corresponding to the characteristic numbers $\lambda_1 < \lambda_2 < \cdots$.

(ii). *Polar case.* When $k(x)$ has both signs and $q(x) \leqq 0$. The system (0.1), (0.2) has an infinite number of normalized characteristic solutions $U_1, U_2, \cdots$ corresponding to the positive characteristic numbers $\lambda_1 < \lambda_2 < \cdots$ and an infinite number $U_{-1}, U_{-2}, \cdots$ corresponding to the negative characteristic numbers $\lambda_{-1} > \lambda_{-2} > \cdots$.

(iii). *Complex case.** When $k(x)$ has both signs and $q(x)$ is positive in at least part of the interval. The system (0.1), (0.2) as in the polar case has two infinite sets of characteristic solutions and characteristic numbers. But, if $q(x)$ is large enough and positive, a finite number of the characteristic numbers $\lambda_1, \cdots, \lambda_{m_1}, \lambda_{-1}, \cdots, \lambda_{-m_2}$ are complex, as are also the characteristic solutions.

Exact theorems concerning the existence of extrema in the various cases are given in §3. In the other sections, however, unless explicit mention is made to the contrary the discussion concerns only the orthogonal case. The argument can generally be carried over to the polar case as is occasionally indicated in the text or a footnote. In the complex case, the problems of the calculus of variations would ordinarily have no meaning.

Intimately related to the differential equation is the calculus of varia tions problem

$$D(u) \equiv \int_0^1 [pu'^2 - qu^2]dx = \min., \tag{0.3}$$

the minimizing function $u(x)$ being subject to the boundary conditions (0.2), the quadratic condition

$$K_0 \equiv \int_0^1 ku^2 dx = 1, \tag{0.4}$$

and the linear conditions

* This case was treated by the author, *Contributions to the study of oscillation properties of the solutions of linear differential equations of the second order*, American Journal of Mathematics, vol. 40 (1918), p. 283.

$$K_i \equiv \int_0^1 kU_i u\, dx = 0 \qquad (i = 1, \cdots, m-1), \tag{0.5}$$

where $U_i(x)$ denotes the solution of the corresponding extremum problem with $i-1$ linear conditions and which may be identified with the solutions $U_i(x)$ of (0.1). The solution of the problem (0.3), (0.2), (0.4), (0.5) is then furnished by $U_m(x)$ satisfying for $\lambda=\lambda_m$ the equation (0.1), to which the Euler condition of all the minimum problems for $m=1, 2, \cdots$ may be reduced. From the equation

$$\int_0^1 (pU_m'^2 - qU_m^2)dx = \lambda_m \int_0^1 kU_m^2 dx,$$

easily derived from (0.1), it will be noted that the value given to $D(u)$ by U_m is λ_m.

The Legendre condition

$$H_{y'y'} = 2p > 0 \tag{0.6}$$

built up after the usual Lagrange method for the function $H \equiv pu'^2 - qu^2 + \lambda ku^2 + \sum_{i=1}^{m-1} 2\mu_i k U_i$, and the Weierstrass condition

$$E \equiv p(u' - \varphi)^2 \geqq 0 \tag{0.7}$$

are satisfied not only by U_m but by all the other admissible solutions $U_{m+1}, U_{m+2}, \cdots$ of the Euler equation (0.1).

The chief interest naturally centered in the Jacobi condition, which excludes the possibility of the point conjugate to $x=0$ in the extended sense lying within the interval 0,1. This condition picks out from the infinite variety of functions U_i automatically satisfying the Euler, Legendre and Weierstrass conditions, that particular one, U_m, which minimizes the integral $D(u)$ under the conditions imposed. This it does by determining the number of oscillations of the function in this interval. In §2 of the present memoir important extensions are made in the discussion of the Jacobi condition.

Although in ordinary problems of the calculus both a maximum and a minimum of the function are usually sought, this has not been the case heretofore in problems of the calculus of variations. This is for the good and sufficient reason that one or other of these is infinite; for example, the maximum in the problem (0.3), (0.2), (0.4), (0.5) is infinite; in fact the conditions (0.6), (0.7) are interpreted to mean that no maximum is possible. In contradiction to these considerations for the ordinary case, some of the problems proposed in this paper possess both maximum and minimum solutions.

Suppose there be added to (0.3), (0.2), (0.4), (0.5) the infinite number of linear conditions

$$\int_0^1 kU_j u\,dx = 0 \qquad (j = s+1, \cdots ; s \geqq m) ; \tag{0.8}$$

as is shown in §3 the minimum is not affected by the addition of these conditions, being furnished by U_m as before. But now a *maximum* of the integral under the same conditions enters and is given by U_s. By computing the Legendre and Weierstrass conditions for the infinitely extended problems it is found that they have respectively the forms (0.6), (0.7) as before; this fits in well with the preconceived notions of a minimum but since these conditions in the same form appear with the maximum problem as well, their significance has, for the moment at least, disappeared. This is perhaps more immediately evident if s is chosen equal to m. The only function orthogonal to U_i for $i=1, \cdots, m-1, m+1, \cdots$, and subject to the conditions (0.2), (0.4) is U_m; this function then furnishes both a maximum and minimum to the integral $D(u)$, while criteria such as the Legendre and Weierstrass should, by all the rules of the game, be different for the two cases. In the treatment of the ordinary problem* the derivation of the Legendre condition is independent of other conditions such as the Jacobi; the same remark may be made concerning the Weierstrass condition as derived by the discoverer. It is noteworthy that the significance of these two criteria as independent conditions has vanished never to return so far as the problems of this paper are concerned. The Weierstrass necessary condition, however, is sometimes deduced on the hypotheses that the Jacobi condition is satisfied in the interval; and in that form, but for the minimum alone, it survives in the problem here discussed. Naturally the Legendre condition, which may be regarded as a less general form of the Weierstrass, must appear in the same rôle. These conditions might well be listed also in some form in any set of sufficient conditions for a minimum of our problem. On the other hand for the maximum there would appear to be no conditions of the usual nature at all possible beyond the Euler equation.

The expectation that the main interest of the new problem would center around the Jacobi condition concerning the conjugate point is fulfilled. For the minimum problem this criterion is placed along side of the Euler as fundamental. The generalized conjugate point must lie outside the interval for a minimum; for the maximum problem proposed it would then follow

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* For example, see Bolza, *Variationsrechnung*. This admirable treatise is a mine of information, and the author wishes to acknowledge his indebtedness to it.

as a condition that the conjugate point lie within the interval. Lying without the interval is a definite criterion and naturally serves as one of a series of sufficient conditions; lying within the interval is a much more shadowy condition. Probably the number of conjugate points existing in the interval is significant, but such a criterion would seem to indicate not much more than the number of steps the maximum problem is removed from the minimum problem.

It appears then that for problems with an infinite number of auxiliary conditions imposed on the function it is to be expected that a generalization of the Euler conditions will retain its importance for both sorts of extrema, and that the generalization of the Jacobi condition will be vital for scrutinizing the various possibilities that present themselves as solutions of the Euler equation. For one sort of extremum the Jacobi condition will probably serve both among the necessary and among the sufficient conditions, while for the other sort its significance will be negative only. On the other hand it is to be expected that the conditions arising as limiting cases of the Weierstrass and Legendre conditions will, for one sort of extrema, be relegated to positions subsidiary to the Jacobi condition, and for the other be dropped out of consideration.

One might go a step further in indicating the breakdown of necessary conditions in problems with an infinite number of auxiliary conditions. In relative maxima and minima of two quadratic forms a necessary and sufficient condition for the existence of an extremum is that one of these forms be definite; which one does not matter. In the present discussion it is not necessary that p be of one sign in order that the integral (0.3) have an extremum. For example consider the problem

$$\int_0^1 (1-2x)y'^2\,dx = \text{extremum},\ y(0) = y(1) = 0,$$

$$\int_0^1 2y^2\,dx = 1,\ \int_0^1 y\sin n\pi x\,dx = 0 \qquad (n = 3,4,\cdots),$$

where only those functions $y(x)$ are to be considered which are continuous and the square of whose derivative is integrable. It may be noted that the only functions satisfying the auxiliary conditions are $c_1 \sin \pi x + c_2 \sin 2\pi x$, $c_1^2 + c_2^2 = 1$. On setting this family of functions in the integral to be made an extremum, there results a quadratic form in the variables c_1, c_2 from which with the relation $c_1^2+c_2^2=1$ the problem may be solved. It would appear that in this case none of the usual necessary conditions have any significance,

not even the Euler condition. In this respect the problem bears some resemblance to the special case of the ordinary isoperimetric problem where the formal solution is a minimizing extremal for the integral involved in the auxiliary condition.

Courant has shown* that if in the problem (0.3), (0.2), (0.4), (0.5) the linear conditions (0.5) be replaced by others more general

$$\int_0^1 V_i u\,dx = 0 \qquad (i = 1,2, \cdots, m-1), \tag{0.9}$$

where $V_i(x)$ are arbitrary continuous functions, and if the minimum (or lower bound) of $D(u)$ be denoted by $D(V_1, \cdots, V_{m-1})$ this minimum cannot be greater than that of the original problem. In other words λ_m is a minimax, that is the maximum of $D(V_1, \cdots, V_{m-1})$ which is itself the minimum of $D(u)$ under the conditions (0.2), (0.4), (0.9). Obviously U_1 furnishes a minimin λ_1, that is a minimum of $D(V_1, \cdots, V_{m-1})$. If there are a denumerable infinity of the conditions (0.9), there can be no minimax, but the minimin is still λ_1.

If the conditions (0.9) are divided into two groups $1, \cdots, l-1$; $l, \cdots, m-1$, the minimum of $D(u)$ will still be a function $D(V_1, \cdots, V_{m-1})$; this may be maximized for $V_1, \cdots, V_{l-1}$, and minimized for $V_l, \cdots, V_{m-1}$, the function U_l giving a minimaximin λ_l.

In the case of both minimum and maximum of $D(u)$ under the conditions (0.2), (0.4), (0.5), (0.8) the Euler equation obtained in a formal manner is

$$(pu')' + qu + \lambda ku - \sum_1^{m-1} \mu_i k U_i - \sum_{s+1}^{\infty} \mu_i k U_i = 0$$

with solutions

$$\alpha u_1(x,\lambda) - \sum_1^{m-1} \frac{\mu_i}{\lambda - \lambda_i} U_i - \sum_{s+1}^{\infty} \frac{\mu_i}{\lambda - \lambda_i} U_i, \tag{0.10}$$

representing an infinity-parameter family of extrema vanishing at $x=0$, the function $u_1(x, \lambda)$ being a solution of the homogeneous equation (0.1) which vanishes at $x=0$. It is shown later that $\mu_i=0$ for all the minimizing and maximizing extremals; in other words these extremals are solutions of the homogeneous equation (0.1).

In discussing the necessity of the Euler equations it may be noted that in the infinite problem the variations which are admitted by the conditions

* R. Courant and D. Hilbert, *Methoden der Mathematischen Physik*, p. 325.

(0.5) must be linearly dependent on $U_m, U_{m+1}, \cdots$, so that any function which cannot be expanded in terms of this partial set of orthogonal functions is barred from consideration. In the problem of this paper admissible variations must be linearly dependent on $U_m, \cdots, U_s$. The family may thus be written

$$\eta = \beta u_1(x,\lambda) + \sum_m^s \beta_i U_i.$$

It is significant that the only function common to this family of admissible variations and the family of extremals (0.10) is the minimizing extremal $u_1(x, \lambda)$.

In many respects the problems of this paper resemble those of relative extrema in quadratic forms involving a finite number or infinite number of variables. The imposition of auxiliary conditions may be regarded as reducing the number of degrees of freedom; when an infinite number of degrees of freedom are taken away there may be a finite or an infinite number remaining. To pursue this notion further let us consider $\sin n\pi x$ as a basic set of functions in terms of which an arbitrary function $u(x)$ vanishing at $x=0$ and $x=1$ is to be expanded in the interval, and set up the corresponding problem of relative extrema in quadratic forms in an infinity of variables. Set

$$u(x) = \sum_1^\infty c_i \sin i\pi x, \quad U_l(x) = \sum_1^\infty d_i^{(l)} \sin i\pi x$$
$$(l = 1,2, \cdots ,m-1 ; s+1, \cdots).$$

The problem is to determine the c's so that the quadratic form

$$(0.11) \quad \int_0^1 \left[p \sum_i ic_i \cos i\pi x \sum_j jc_j \cos j\pi x - q \sum_i c_i \sin i\pi x \sum_j c_j \sin j\pi x \right] dx \equiv \sum_{ij} e_{ij} c_i c_j$$

is an extremum under the quadratic condition

$$(0.12) \quad \int_0^1 \left[k \sum_i c_i \sin i\pi x \sum_j c_j \sin j\pi x \right] dx \equiv \sum_{ij} g_{ij} c_i c_j = 1,$$

and the infinite number of linear conditions

$$(0.13) \quad \int_0^1 \left[k \sum_i d_i^{(l)} \sin i\pi x \sum_j c_j \sin j\pi x \right] dx \equiv \sum_{ij} h_{ij} c_i d_j^{(l)} = 0.$$

This leads formally to the problem of finding an extremum for

$$\sum_{ij} e_{ij}c_ic_j + \lambda \sum_{ij} g_{ij}c_ic_j + \sum_l \mu_l \sum_{ij} h_{ij}c_id_j^{(l)},$$

subject to the conditions (0.12), (0.13) and on differentiation with regard to the c's and the μ's gives rise to the linear equations

$$\sum_j (e_{ij} + \lambda g_{ij})c_j + \sum_l \mu_l \sum_j h_{ij}d_j^{(l)} = 0 \qquad (i = 1, 2, \cdots) \tag{0.14}$$

together with (0.13). In order that this infinity of linear homogeneous equations in c's and μ's have a solution it is necessary that λ be a root of an infinite determinant consisting of four groups, each of infinite extent in both directions. This may be written

(0.15)

$$\begin{vmatrix} e_{11} + \lambda g_{11} & e_{12} + \lambda g_{12} & \cdots & \sum_j h_{1j}d_j^{(1)} & \sum_j h_{1j}d_j^{(2)} & \cdots \\ e_{21} + \lambda g_{21} & e_{22} + \lambda g_{22} & \cdots & \sum_j h_{2j}d_j^{(1)} & \sum_j h_{2j}d_j^{(2)} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_j h_{1j}d_j^{(1)} & \sum_j h_{2j}d_j^{(1)} & \cdots & 0 & 0 & \cdots \\ \sum_j h_{1j}d_j^{(2)} & \sum_j h_{2j}d_j^{(2)} & \cdots & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

The quadratic condition (0.12) fixes the multiplicative constant involved in the solution of the homogeneous equations. Since from the method of definition, $e_{ij}=e_{ji}$, $g_{ij}=g_{ji}$, $h_{ij}=h_{ji}$ the determinant is symmetric. Since it is known in advance (§3) that both the maximum and minimum problems have solutions, the infinite-bordered determinant must have $s-m+1$* roots λ_i. For these values the solutions of the linear equations (0.12), (0.14) furnish the various sets of c's which give not only the solutions U_s, U_m of the problems but also the other functions $U_{m+1}, \cdots, U_{s-1}$.

In studying these problems of maximizing and minimizing the quadratic form (0.11) under the quadratic condition (0.12) and the infinite number of linear conditions (0.13), the question naturally presents itself as to

* For the minimum problem in the polar case the interesting situation develops that the determinant corresponding to (0.15) has an infinite number of roots λ, each of which is known in advance, and for each of which the equations (0.14) have solutions.

what is the condition (analogous in some respects to the Jacobi criterion in the calculus of variations) which picks out, in one case, U_s, and in the other, U_m, from the various possibilities $U_m, \cdots, U_s$. For the same problem in a finite number of variables the author has derived this condition*; that discussion suggests an analogous theorem here.

If instead of sin $n\pi x$ the functions U_i are used as basic system the treatment is much simplified. As may be seen from the discussion in §3 all terms of the determinant (0.15) vanish except those in the main diagonal of each of the three non-zero divisions.

To indicate the connection† with the theory of integral equations, denote by $G(x, \xi)$ the Green's function of the differential expression

$$M(u) \equiv (pu')' + qu, \tag{0.16}$$

corresponding to the boundary conditions (0.2). Then the integral equation

$$u(x) = \lambda \int_0^1 k(\xi)G(x,\xi)u(\xi)d\xi \tag{0.17}$$

has the same solutions as the system (0.1), (0.2).

On setting $M(u)=h(x)$, we have from the known properties of the Green's function

$$u(x) = -\int_0^1 G(x,\xi)h(\xi)d\xi. \tag{0.18}$$

On the other hand, integration by parts gives

$$D(u) = -\int_0^1 uM(u)\,dx = \int_0^1\int_0^1 G(x,\xi)h(x)h(\xi)dx\,d\xi. \tag{0.19}$$

Thus the discussion of the extrema of the integral $D(u)$ is reduced to that of the integral on the right of (0.19). If we multiply (0.17) by $k(x)$ $u(x)$ and integrate, we obtain the formula

$$K_0(x) \equiv \int_0^1 ku^2\,dx = \lambda\int_0^1\int_0^1 k(x)k(\xi)G(x,\xi)u(x)u(\xi)dx\,d\xi \equiv \lambda R(u), \tag{0.20}$$

and hence from (1.10) we have, when U_i is a characteristic function,

* *Relative extrema of pairs of quadratic and hermitian forms*, these Transactions, vol. 26, p. 491.

† I am indebted to my colleague, Professor J. Tamarkin, for suggestions concerning the methods used in connection with these expansions.

$$D(U_i) = \lambda_i K_0(U_i) = \lambda_i^2 R(U_i).$$

When $q \leqq 0$ the integral D is positive and hence the integral R is also positive.

In §3 the three integrals $D(u)$; $K_0(u)$, and $R(u)$ are discussed in regard to relative maxima and minima under an infinite number of linear auxiliary conditions.

The Jacobi condition as discussed in §6 concerns the non-vanishing of an infinite determinant involving integrals. When any finite number of conditions are dropped from the set of linear conditions (0.8), the infinite determinant corresponding to the resulting problem has again no zero in the interval 0, 1, and it is a curious fact that its ratio to the original is a decreasing function throughout the interval.

A portion of the discussion in this paper is too formal, omitting much in the way of justification of infinite processes. Since, however, the extrema actually exist, the main argument is correct and the briefer treatment has its advantages.

It may be noted further that the linear character of all except one of the auxiliary conditions renders the treatment much simpler than it would be in the general case. In particular the analogons of the Legendre and Weierstrass criteria and of the Hamilton function and Hilbert integral have very simple forms.

The results of this paper as here given for the simple boundary conditions (0.2) may be extended without difficulty to more complicated cases. The treatment as given for one independent variable may be readily generalized to regions of two or more dimensions. With the exception of the process of taking the derivatives of the quotients of the determinants arising in the discussion of the Jacobi condition, all notions and methods go over almost without change to the more general problem. The interpretation of the Jacobi condition in terms of oscillation theorems for two or more independent variables, however, is obscure and difficult and has not been worked out.

1. Preliminary theorems and formulas

In this section we shall assemble some fundamental formulas for later reference and shall review some of the considerations of the paper* which treats the case of a finite number of auxiliary conditions.

Basic for the argument is the self-adjoint differential equation of the second order

$$L(u) \equiv \big(p(x)u'(x)\big)' + q(x)u(x) + \lambda k(x)u(x) = 0, \tag{1.1}$$

* Loc. cit., Mathematische Annalen, vol. 68, p. 269

with the boundary conditions

$$u(0) = u(1) = 0, \tag{1.2}$$

where $p>0$, and where p, q and k are analytic functions* of x in the interval 0, 1 considered.

The general solution $\alpha u_1(x, \lambda)+\beta u_2(x, \lambda)$ of (1.1) contains two arbitrary constants besides the parameter λ. Since the discussion of this paper concerns only the family through $x=0$, u_1 may be chosen so as to vanish there and the solution may then be written

$$u = \alpha u_1(x,\lambda) \tag{1.3}$$

where it is assumed for the sake of uniformity and without loss of generality that $\alpha>0$, $u'(0, \lambda)>0$. As $|\lambda|$ increases all the zeros of $u_1(x,\lambda)$ (except that at $x=0$) move to the left.

As noted in the Introduction, there are two important cases connected with the problems of the calculus of variations. In the *orthogonal case* there is an infinite set

$$U_1, U_2, \cdots \tag{1.4}$$

of solutions of (1.1) (1.2) and in the *polar case* there are two such sets

$$U_1, U_2, \cdots; \qquad U_{-1}, U_{-2}, \cdots. \tag{1.5}$$

Solutions can be considered orthogonalized and normalized:

$$\int_0^1 kU_iU_j\,dx = 0(i \neq j)\ ;\int_0^1 kU_i^2dx = 1 \quad \left[\int_0^1 kU_{-i}^2dx = -1\right]. \tag{1.6}$$

For the orthogonal case, the equation (1.1) is the Euler condition for the calculus of variations problem

$$D(u) \equiv \int_0^1 (pu'^2 - qu^2)dx = \text{min.}, \quad u(0) = u(1) = 0, \tag{1.7}$$

subject to the quadratic auxiliary condition

$$K_0 \equiv \int_0^1 ku^2dx = +1. \tag{1.8}$$

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* The main features of the discussion can be carried through under much less stringent conditions.

For, on setting

$$(1.9)\qquad v_0(x) = \int_0^x ku^2dx, \quad \text{and hence} \quad v_0' - ku^2 = 0,\ v_0(0) = 0,\ v_0(1) = 1,$$

an application of the Lagrange method transforms the relative minimum problem into that of finding an absolute minimum of the integral

$$\int_0^1 [pu'^2 - qu^2 + \lambda(v_0' - ku^2)]dx$$

having for Euler condition the equation (1.1).

The solution of the minimum problem must then be found amon g(1.4); from the formulas easily derived from (1.1),

$$(1.10)\qquad \int_0^1 (pu'^2 - qu^2)dx = \lambda \int_0^1 ku^2dx,$$

it follows that the minimum value is one of the λ's. Since all the other conditions of the minimum problem are satisfied by any of the functions $U_1, U_2, \cdots$, it must be the Jacobi criterion alone which determines that particular one, U_1, having no zero within the interval.

If the extremum problem (1.7), (1.8) is changed by the addition of the linear conditions

$$(1.11)\qquad K_i \equiv \int_0^1 kU_i\, udx = 0 \qquad (i = 1, 2, \cdots, m-1),$$

the solution U_1 is barred from consideration. On setting

$$(1.12)\qquad v_i = \int_0^x kU_i\, udx, \quad \text{and hence} \quad v_i' - kU_i u = 0,\ v_i(0) = v_i(1) = 0,$$

and considering the problem of minimizing the integral

$$\int_0^1 \left[pu'^2 - qu^2 + \lambda(v_0' - ku^2) + \sum_1^{m-1} 2\mu_i(v_i' - kU_i u)\right]dx,$$

the Euler equation takes the non-homogeneous form

$$(1.13)\qquad (pu')' + qu + \lambda ku + \sum_1^{m-1} \mu_i kU_i = 0$$

with solutions

$$(1.14)\qquad u = \alpha u_1(x,\lambda) - \sum_1^{m-1} \frac{\mu_i U_i}{\lambda - \lambda_i},$$

as may be proved by substitution. This family of curves may be used as the extremals of the problem. It is possible to show that for the minimizing extremal of this family all the μ's are zero. For, on setting the value of u from (1.14) in (1.11), using the boundary conditions and the relations (1.6), we find that

$$(1.15)\quad 0 = v_i(1) = -\frac{\mu_i}{\lambda-\lambda_i}\int_0^1 kU_i^2\,dx = -\frac{\mu_i}{\lambda-\lambda_i}\quad (i = 1, \cdots, m-1),$$

from which it follows that $\mu_i=0$. The differential equation of the minimizing extremals is thus reduced from (1.13) to (1.1). The Jacobi condition selects the solution which is in this case U_m with $m-1$ zeros within the interval.

It should be noted that for some purposes, such as the Jacobi condition, it is well to interpret the family of extremals as being in higher dimensional space. By adding to the two dimensions xu of (1.14), a third v_0 given by (1.9) and $m-1$ more v_i given by (1.12), the extremals may be considered to be curves in the $(m+2)$-dimensional xuv_0v_i space.

For the polar case, there are two sets of calculus of variations problems for which the equation (1.1) is the Euler condition. One is precisely that of the formulas (1.7) to (1.15); the other is set up by replacing the quadratic condition (1.8) by $K_0=-1$.

Let $f(x)$ be any function which vanishes at $x=0$ and $x=1$ and which can be represented in the form

$$f(x) = \int_0^x \phi(x)dx,$$

where $\phi(x)$ is integrable together with its square. It is known* that $f(x)$ can be expanded in an absolutely and uniformly convergent series (in both the orthogonal and the polar case):

$$(1.16)\qquad f(x) = \sum f_iU_i(x),\quad f_i = \operatorname{sign}\lambda_i\int_0^1 kfU_i\,dx,$$

the summation being taken over all the characteristic values. Substituting here $f(\xi)=G(x,\xi)$ and observing from (0.17) that

$$(1.17)\qquad U_i(x) = \lambda_i\int_0^1 k(\xi)G(x,\xi)U_i(\xi)d\xi,$$

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* J. Tamarkine, *Problème du développement d'une fonction arbitraire en séries de Sturm-Liouville*, Comptes Rendus, vol. 156 (1913), pp. 1589–1591; L. Lichtenstein, *Zur Analysis der unendlichvielen Variabeln*, Rendiconti del Circolo Matematico di Palermo, vol. 38 (1914), pp. 113–166.

it is readily seen that

$$G(x,\xi) = \sum \frac{U_i(x)U_i(\xi)}{|\lambda_i|}, \tag{1.18}$$

the series being absolutely and uniformly convergent.

If $h(x)$ be an integrable function, on multiplying (1.18) by $h(x)\ h(\xi)$ and integrating we get the bilinear formula

$$\int_0^1 \int_0^1 G(x,\xi)h(x)h(\xi)dxd\xi = \sum \frac{h_i^2}{|\lambda_i|}; \quad h_i = \int_0^1 hU_i\,dx. \tag{1.19}$$

Let us now identify $h(x)$ with $M(u)$ as defined in (0.16). If $u(x)$ is a function whose first derivative is absolutely continuous in 0,1, then $u''(x)$ exists almost everywhere and is integrable. Hence from (0.19) and (1.19) we have $D(u) = \sum (h_i^2/|\lambda_i|)$, and since from (1.19), (1.17), and (0.18)

$$h_i = \lambda_i \int_0^1 \int_0^1 h(x)\ k(\xi)G(x,\xi)U_i(\xi)dxd\xi = -\lambda_i \int_0^1 k(\xi)u(\xi)U_i(\xi)d\xi,$$

this may be written

$$D(u) = \sum |\lambda_i|\, c_i^2, \quad c_i = \operatorname{sign} \lambda_i \int_0^1 k(x)u(x)U_i(x)dx. \tag{1.20}$$

We note also that from (1.16) there follows

$$K_0(f) \equiv \int_0^1 kf^2dx = \sum \operatorname{sign} \lambda_i f_i^2; \ K_0(u) = \sum \operatorname{sign} \lambda_i c_i^2. \tag{1.21}$$

Some relations between the various solutions of (1.1) are important and will now be developed. It may be noted that the function $u(x, \lambda)$ satisfies the equation

$$\left(p\frac{\partial u'}{\partial \lambda}\right)' + (q + \lambda k)\frac{\partial u}{\partial \lambda} + ku = 0, \quad \frac{\partial u(0)}{\partial \lambda} = 0. \tag{1.22}$$

On multiplication of (1.1) for the characteristic number λ_m and solution U_m by $\partial u/\partial\lambda$ and of (1.22) by $-U_m$, addition and integration, we obtain the relation

$$U_m'\frac{\partial u}{\partial \lambda} - U_m\frac{\partial u'}{\partial \lambda} = \frac{\lambda - \lambda_m}{p}\int_0^x kU_m\frac{\partial u}{\partial \lambda}dx + \frac{1}{p}\int_0^x kU_m u\,dx, \tag{1.23}$$

which for the special case $u = U_1$ and $m = 1$ becomes

$$p\left(u'\frac{\partial u}{\partial \lambda} - u\frac{\partial u'}{\partial \lambda}\right) = \int_0^x ku^2dx. \tag{1.24}$$

If U_m, U_l denote any two of the family (1.4) (or (1.5)) corresponding to the parameters λ_m, λ_l it may be proved in a similar manner that they satisfy the identity

$$U_m' U_l - U_m U_l' = \frac{\lambda_l - \lambda_m}{p} \int_0^x k U_m U_l \, dx. \tag{1.25}$$

If η is a continuous function vanishing at 0 and x_1, on multiplication of $L(u)=0$ by η and integration by parts there results a relation $D(u, \eta) = \lambda K_0(u, \eta)$ between the polar forms $D(u, \eta) \equiv \int_0^{x_1}(pu'\eta' - qu\eta)dx$, $K_0(u, \eta) \equiv \int_0^{x_1} ku\eta dx$. Further, let η be an allowable variation in the interval 0, x_1 for the problem (1.7), (1.8), (1.11); that is, let

$$K_0(u, \eta) = 0, \quad K_i \equiv \int_0^{x_1} kU_i\eta dx = 0 \, ;$$

then $D(u, \eta) = 0$.

2. An extension of the finite problem. The Jacobi condition and its interpretation

Using the method of the earlier paper* let us pursue considerably further than was there necessary the question of the Jacobi condition for the finite case. Consider the new problem of a minimum of $D(u)$ under the boundary conditions $u(0) = u(1) = 0$, the quadratic condition $K_0 = 1$(1.8) and the two sets of linear conditions

$$K_i \equiv \int_0^1 kU_i u dx = 0, \quad i = 1, \cdots, m-1\,; \; i = s+1, \cdots, l; \; s \geqq m. \tag{2.1}$$

The addition of the second set of (2.1) cannot decrease the minimum; that it is not increased is readily seen by noting that the function furnishing the minimum for the first set (2.1) only is U_m and that this function also satisfies the second set. That the minimum is λ_m furnished by U_m may also be proved in a manner analogous to Theorem IV of §3. In other words the second set of conditions (2.1) affects the problem only formally. The Euler equation takes the form

$$(pu')' + qu + \lambda ku + \sum_1^{m-1} \mu_i kU_i + \sum_{s+1}^{l} \mu_i kU_i = 0, \tag{2.2}$$

with the $(m+l-s+1)$-parameter family of solutions through the origin

* Loc. cit., Mathematische Annalen, vol. 68, p. 289.

$$u = \alpha u_1(x,\lambda) - \sum_1^{m-1} \frac{\mu_i U_i}{\lambda - \lambda_i} - \sum_{s+1}^{l} \frac{\mu_i U_i}{\lambda - \lambda_i}, \tag{2.3}$$

where $\alpha u_1(x, \lambda)$ is defined as in (1.3). By a method similar to that used in §1, the plane family of extremals (2.3) may be replaced by an $(m+l-s+1)$-parameter $(\alpha\lambda\mu_i)$-family in $(m+l-s+2)$-dimensional (xuv_0v_i)-space

$$\begin{aligned} u &= \alpha u_1(x,\lambda) - \sum \mu_i \frac{U_i}{\lambda - \lambda_i}, \quad v_0 = \int_0^x k\left(\alpha u_1 - \sum \frac{\mu_i U_i}{\lambda - \lambda_i}\right)^2 dx, \\ v_j &= \int_0^x k\left(\alpha u_i - \sum \frac{\mu_i U_i}{\lambda - \lambda_i}\right) U_j \, dx \quad (j = 1, \cdots, m-1; s+1, \cdots, l), \end{aligned} \tag{2.4}$$

and passing through the origin $(0, 0, \cdots, 0)$. The summation over i, here as hereafter, is supposed to extend through the range $1, \cdots, m-1$, $s+1, \cdots, l$. It may be shown as in §1 that in this family (2.4) there is imbedded the space curve corresponding to the minimizing extremal $u = U_m(x, \lambda_m)$ and for which $\mu_i = 0$, $\lambda = \lambda_m$. Geometrically interpreted, the Jacobi condition demands that within the x-interval 0, 1 this space curve be not cut by any of its neighbors. This is equivalent to saying that the $m+l-s+1$ homogeneous equations in as many unknowns

$$\begin{aligned} &\delta\lambda\left[\alpha \frac{\partial u_1(x, \lambda_m)}{\partial \lambda}\right] + \delta\alpha[u_1(x,\lambda_m)] - \sum \delta\mu_i\left(\frac{U_i(x)}{\lambda_m - \lambda_i}\right) = 0, \\ &2\alpha^2\delta\lambda \int_0^x k u_1 \frac{\partial u_1}{\partial \lambda} dx + 2\alpha\delta\alpha \int_0^x k u_1^2 \, dx - \sum \frac{2\alpha\delta\mu_i}{\lambda_m - \lambda_i} \int_0^x k u_1 U_i dx = 0, \\ &\alpha\delta\lambda \int_0^x k U_j \frac{\partial u_1}{\partial \lambda} dx + \delta\alpha \int_0^x k u_1 U_j dx - \sum_i \left(\frac{\delta\mu_i}{\lambda_m - \lambda_i}\right) \int_0^x k U_i U_j dx = 0 \end{aligned} \tag{2.5}$$

have no solution for $0 < x < 1$. The value of x next after $x = 0$ for which these equations hold is called the conjugate point in the extended sense. And the Jacobi condition demands that this conjugate point lie beyond the point $x = 1$. Now an infinite set $U_m, \cdots, U_s, U_{l+1}, \cdots$ of characteristic solutions of (1.1) satisfy all the others of the set of sufficient conditions; hence it is the Jacobi condition alone which selects U_m as the minimum. It will later be shown that the condition implies that u_1 vanish $m-1$ times in the interval, thus identifying it with U_m except for a constant multiplier.

On the analytic side, the Jacobi condition concerns the sign of the second variation. For the purpose of calculating the second variation, we may take the integral in the form

$$\int_0^1 [pu'^2 - qu^2 + \lambda_m(v_0' - ku^2) + \sum 2\mu_i(v_i' - kU_i u)]dx.$$

The admissible variation η is subject to the restrictions $\eta(0)=\eta(1)=0$ and

$$\int_0^1 ku\eta dx = 0, \quad \int_0^1 kU_i\eta dx = 0 \; (i = 1, \cdots, m-1; s+1, \cdots, l). \tag{2.6}$$

After the usual computation we find

$$\delta^2 D = \epsilon^2 \int_0^1 (p\eta'^2 - q\eta^2 - \lambda_m k\eta^2)dx \tag{2.7}$$

which by integration by parts and addition of multiples of the linear terms (2.6) becomes

$$\delta^2 D = \epsilon^2 \int_0^1 - \eta[(p\eta')' + q\eta + \lambda_m k\eta + \alpha\delta\lambda ku + \sum \mu_i kU_i]dx. \tag{2.8}$$

Consider the expression inside the brackets of this integrand; it will vanish if for η we substitute the left hand side of the first line of (2.5), as can be proved by substitution and use of (1.1) and (1.22) and remembering that $\mu_i = \delta\mu_i$. It follows that if $x=1$ is the point conjugate to $x=0$, the second variation $\delta^2 D$ may be made zero by giving to η this value.

A similar argument may be applied to any interval $0, x_1$, where x_1 is the point conjugate to 0. If x_1 is within the interval 0, 1, and if η is an admissible variation over $0, x_1$, we may set $\eta=0$ in the interval x_1, 1. In that case the conditions (2.6) still hold and the second variation may still be written in the form (2.8) and may be made to vanish by the same device.

The original minimum problem for u can be put into essentially the same form as (2.7) (2.6) with a proper quadratic restriction. Hence the minimum for $D(\eta)/K_0(\eta)$ is λ_m furnished by $\eta = U_m(x)$ which is an analytic function. Any variation which is zero in a part of the interval cannot be analytic and hence cannot furnish the minimum for (2.7). In that case $\delta^2 D$ can be made negative, which indicates that the point conjugate to $x=0$ cannot lie within the interval.

For the sake of definiteness, let us choose $m=2$, $s=3$, $l=4$ and proceed to set up in detail the Jacobi condition. The Jacobi determinant of (2.5), apart from a constant factor, is

$$
(2.9)\qquad D_{14}(x,\lambda) = \begin{vmatrix} \dfrac{\partial u_1}{\partial\lambda} & u_1 & U_1 & U_4 \\ \int_0^x ku_1\left(\dfrac{\partial u_1}{\partial\lambda}\right)dx & \int_0^x ku_1^2\,dx & \int_0^x ku_1U_1dx & \int_0^x ku_1U_4dx \\ \int_0^x kU_1\left(\dfrac{\partial u_1}{\partial\lambda}\right)dx & \int_0^x kU_1u_1dx & \int_0^x kU_1^2dx & \int_0^x kU_1U_4dx \\ \int_0^x kU_4\left(\dfrac{\partial u_1}{\partial\lambda}\right)dx & \int_0^x kU_4u_1dx & \int_0^x kU_4U_1dx & \int_0^x xU_4^2dx \end{vmatrix}
$$

and for a minimum the Jacobi condition asserts that this can have no zero within the interval. It vanishes at $x=0$, but not at $x=1$ since at that point its value is $\partial u_1/\partial\lambda$ and from (1.24) and (1.6) it is evident that not both u and $\partial u/\partial\lambda$ can vanish at $x=1$.

Add to the conditions of this special problem the further one

$$
\int_0^1 kU_3u\,dx = 0.
$$

The solution is still U_2, but in place of $D_{14}(x, \lambda)$ there is a five-rowed determinant $D_{134}(x, \lambda)$, which is obtained by inserting between the third and fourth rows of (2.5) a new row similar to these except that U_3 replaces U_1 or U_4 and between the third and fourth column a new column in similar fashion. Schematically this new determinant, which must not vanish within the interval, may be expressed as follows:

$$
(2.10)\qquad D_{134}(x,\lambda) = \begin{vmatrix} \dfrac{\partial u_1}{\partial\lambda} & u_1 & U_1 & U_3 & U_4 \\ a_2 & a_{22} & a_{23} & a_{24} & a_{25} \\ a_3 & a_{32} & a_{33} & a_{34} & a_{35} \\ a_4 & a_{42} & a_{43} & a_{44} & a_{45} \\ a_5 & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix}.
$$

Before proceeding further with the main argument, let us prove a fundamental lemma, the compact form of the proof of which is due to my colleague, Professor H. P. Manning.

LEMMA. *Given two determinants, D_m and D_{m+1}, of the mth and $(m+1)$th orders, respectively, the first being a first minor of the second,*

$$
\begin{vmatrix} w & w_1 & \cdots & w_{m-1} \\ a_1 & a_{1,1} & \cdots & a_{1,m-1} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m-1} & a_{m-1,1} & \cdots & a_{m-1,m-1} \end{vmatrix}, \qquad \begin{vmatrix} w & w_1 & \cdots & w_m \\ a_1 & a_{1,1} & \cdots & a_{1,m} \\ \cdot & \cdot & \cdots & \cdot \\ a_m & a_{m,1} & \cdots & a_{m,m} \end{vmatrix}
$$

and the terms being subject to the following conditions:

$$a_{ij} = a_{ji}, \quad ww_i' - w'w_i = a_{1i}f(x) + l_ia_i, \quad w_iw_j' - w_j'w_i = (l_j - l_i)a_{ij},$$

and such that the derivatives of the terms of any row other than the first form multiples of the first. Denoting by $\alpha, \alpha_1, \cdots, \alpha_{m-1}$ *the cofactors of the first row of* D_{m-1} *and by* $A, A_1, \cdots, A_m$ *those of the first row of* D_m *and for completeness of notations setting* $\alpha_m = 0$, *then*

$$\frac{d}{dx}\frac{D_m}{D_{m-1}} = \frac{l_mA_m^2}{D_{m-1}^2}\,.$$

From the theory of determinants the following identities may be written down:

$$(2.11)\quad a_j\alpha + \sum_{i=1}^{m}\alpha_ia_{ji} = 0 \ (j = 1, \cdots, m-1)\ ; \ a_m\alpha + \sum_{i=1}^{m}\alpha_ia_{mi} = A_m\ ;$$

$$(2.12)\quad a_jA + \sum_{i=1}^{m}A_ia_{ji} = 0 \qquad (j = 1, \cdots, m).$$

Since by hypothesis the determinants formed by replacing any row except the first by its derivative are zero, the numerator of the derivative of the quotient of the determinants can be written in the form of a single determinant of the second order

$$(2.13)\quad \begin{vmatrix} w\alpha + w_1\alpha_1 + \cdots + w_m\alpha_m & w'\alpha + w_1'\alpha_1 + \cdots + w_m'\alpha_m \\ wA + w_1A_1 + \cdots + w_mA_m & w'A + w_1'A_1 + \cdots + w_m'A_m \end{vmatrix}.$$

In expanding (2.13) the terms involving either w or w' may be written $\sum(ww_i' - w'w_i)(\alpha A_i - A\alpha_i)$. By hypothesis this becomes

$$\begin{aligned} &f(x)\sum a_{1i}(\alpha A_i - A\alpha_i) + \sum a_il_i(\alpha A_i - A\alpha_i) \\ &\qquad \equiv f(x)\left[\alpha(a_1A + \sum a_{1i}A_i) - A(a_1\alpha + \sum a_{1i}\alpha_i)\right] + \sum l_ia_i(\alpha A_i - A\alpha_i) \end{aligned}$$

which by (2.11) and (2.12) becomes

$$(2.14)\qquad \sum a_il_i(\alpha A_i - A\alpha_i).$$

The other terms of the expansion of (2.13) are $\sum_{ij}(w_iw_j' - w_i'w_j)(\alpha_iA_j - \alpha_jA_i)$ $(j > i)$ which by hypothesis may be written

$$\begin{aligned} &\sum_{ij}(l_j - l_i)a_{ij}(\alpha_iA_j - \alpha_jA_i) && (j > i) \\ &= \sum_{ij}\left[l_j\alpha_iA_j + l_i\alpha_jA_i - l_j\alpha_jA_i - l_i\alpha_iA_j\right]a_{ij} && (j > i) \\ &= \sum_j l_jA_j\sum_i\alpha_ia_{ij} - \sum_j l_j\alpha_j\sum_i A_ia_{ij} && (i \text{ and } j \text{ independent}). \end{aligned}$$

Adding this last expression to (2.14) we have for the numerator of the determinant

$$\sum_j l_j A_j\left(\alpha a_j + \sum_i \alpha_i a_{ij}\right) - \sum_j l_j \alpha_j\left(A\alpha_j + \sum_i A_i a_{ij}\right).$$

By (2.11) and (2.12) this reduces to $l_m A_m^2$. The derivative is thus $l_m A_m^2/D_{m-1}^2$ and has the sign of l_m.

Further it may be noted that if from D_m we pick out another minor E_{st} by leaving out any row (the sth) except the first and any column (the tth) except the first, the same argument holds and we find that

$$\frac{d}{dx}\frac{D_m}{E_{st}} = l_s\frac{A_s A_t}{E_{st}^2}.$$

The argument may also be applied in a formal fashion when m is infinite.

Returning now to the main discussion it is possible to write down the derivative with regard to x of the quotient of (2.10) and its first minor (2.9). As may be seen from (1.23), (1.24), (1.25), the conditions of the lemma are satisfied by the determinants (2.9), (2.10).

Hence we have

$$\text{(2.15)} \qquad \frac{d}{dx}\frac{D_{14}}{D_{134}} = \frac{\lambda_2 - \lambda_3}{pD_{134}^2}\alpha_3^2 < 0,$$

where α_3 is the cofactor of U_3 in (2.10).

Now D_{134} and D_{14} vanish at $x=0$, do not vanish in the interval, and have at $x=1$ the same value

$$\left.\frac{\partial u_1(x, \lambda)}{\partial\lambda}\right|_{x=1,\lambda=\lambda_3}$$

which is positive as may be noted from (1.24), since $U_3(1)=0$ and $U_3' > 0$, this being the second zero beyond $x=0$ for this function.

The formula (2.15) indicates that the roots of D_{134} and D_{14} separate each other and since it may be shown as in §7 of the paper cited that at $x=0$ the determinant D_{134} has the higher order of zero, the quotient D_{14}/D_{134} starts at $x=0$ with a value $+\infty$ and having a value 1 at $x=1$, vanishes at the first zero of D_{14} which must lie before that of D_{134}.

If D_1 denotes the determinant obtained by omitting the last row and column from (2.9), the same argument shows that

$$\frac{d}{dx}\frac{D_1}{D_{14}} = \frac{\lambda_3 - \lambda_4}{p(x)}(\text{function})^2 < 0.$$

D_1 vanishes at $x=0$ of lower order than D_{14} and has at $x=1$ the same value; the next root of D_1 must then lie before that of D_{14}.

In descending one step further in the order of the determinant, the argument is somewhat different and coincides with that of the earlier paper. The formal process of finding by means of the Lemma the derivative of the quotient of two determinants is the same but since in all cases the sign of the result depends on $\lambda_2-\lambda_i$, the derivative is negative when any condition of the second set of (2.1) is involved and positive when all of these are omitted. The ratio of the determinants at $x=0$ is $+\infty$ in the first case and $-\infty$ in the other.

When the last row and column of D_1 are omitted and the remaining two-rowed determinant denoted by D, it was shown in the earlier paper* (and also follows from the discussion here) that

$$\frac{d}{dx}\frac{D}{D_1}=\frac{\lambda_2-\lambda_1}{p(x)}(\text{function})^2>0,$$

and further that

$$\frac{d}{dx}\frac{u_1}{D}=\frac{1}{p(x)}(\text{function})^2>0,$$

and from these facts that u_1 has precisely one zero between $x=0$ and $x=1$. The Jacobi condition is thus vital in the final handling of this calculus of variations problem. Its place among the necessary conditions and among the sufficient conditions is a fundamentally important one.

The determinant D_{14} is obtained from D_{134} by omitting the fourth row and fourth column; if a different four-rowed minor be selected from D_{134} by omitting any row except the first and any column, a formula for the derivative of the quotient of it by D_{134} may be obtained in the same manner.† If the minor selected be symmetrically placed with regard to the main diagonal, the derivative will involve the square of a minor as in (2.8); if it is not symmetrically placed this square is replaced by the product of two different minors. This process may be repeated step by step until one arrives at u_1. The ratio of D_{134} (or of u) to any minor of any order symmetrical to the diagonal is a function of x monotone in the interval 0, 1 and one may descend from D_{134} to u_1 by ladders different from that used above; but in each case the argument determines the exact number of zeros of u_1.

* Loc. cit., Mathematische Annalen, vol 68, p. 269.

† Cf. the sequel of the lemma.

Returning to the other end of the series of determinants, if an extra condition is imposed on u_1 so as to give a six-rowed determinant D_{134l}, including D_{134} as a first minor, we have

$$\frac{d}{dx}\frac{D_{134}}{D_{134l}} = (\lambda_3 - \lambda_l)(\text{function})^2 < 0.$$

For the various functions, the first zeros beyond $x=0$ lie in the following order from left to right: u_1, D, D_1, D_{14}, D_{134}, D_{134l}.

For the general problem with conditions (2.1) the essential facts may be formulated in a fashion similar to that of the special case selected. If the second set is deleted, the determinant $D_{1 \dots m-1}$ has no zero within 0, 1 while $D_{1 \dots m-2}$ has one, $D_{1 \dots m-3}$ has two and u_1 has $m-1$. But the addition of any group of one or more (and in any order) of the second set (which is more or less supernumary to the problem) gives a determinant with no zero within the interval. The imposition of another condition moves further to the right the zero of the determinant, and this continues step by step until as many conditions are imposed as is desired. It is striking that determinants of integrals of any desired order and with no zero in the interval 0, 1 can be built up in this simple fashion. It is also noteworthy that the ratio of the determinant or of any minor symmetrically placed with regard to the main diagonal to any other minor contained in it and also symmetrically placed is a function of x monotone in the interval, provided only that the latter contains the term u_1.

It may further be remarked that the above discussions apply not only when there are two groups of linear conditions $K_i=0$, each with consecutive subscripts, but also when these conditions are taken at random. The minimum is furnished by U_p where p is the smallest integer not included among the i's; the Jacobi condition admits of interpretation as in the case discussed. It is also immediately evident that a minimum would exist if i ran over some sequence not including all the integers but with infinity as a limit. The argument of this section paves the way for the extension of the theory to the infinite case.

3. The existence of extrema

In §1 a single sequence of functions $U_1, U_2, \cdots$ was defined in the orthogonal case and a double sequence $U_1, U_2, \cdots; \cdots, U_{-2}, U_{-1}$ was defined in the polar case as solutions of the differential equation

$$L(u) \equiv (pu')' + qu + \lambda ku = 0 \tag{3.1}$$

under the boundary conditions

$$u(0) = u(1) = 0. \tag{3.2}$$

The theorems of the present section concerning these functions fall into two groups according as the orthogonal ($k(x)$ one sign) or the polar case ($k(x)$ both signs) is considered. For the polar case it is possible (by the addition of an infinity of linear conditions imposed on the functions U_{-i}) to establish results in nature similar to those of Theorem I; but the principles involved are sufficiently illustrated by the less complicated formulation here given. For the sake of simplicity in the polar case a further hypothesis is made that all the characteristic solutions are real; this will be the case, for example, if $q(x) \leqq 0$.

The relative extrema here discussed concern three integrals the relations of each of which to the differential equation (3.1) have been discussed in the Introduction. These are

$$D(u) \equiv \int_0^1 (pu'^2 - qu^2)dx, \quad p > 0 \; ; \; K_0(u) \equiv \int_0^1 ku^2 dx,$$

$$R(u) \equiv \int_0^1 \int_0^1 k(x)k(\xi)\,G(x,\xi)u(x)u(\xi)dxd\xi,$$

where $G(x, \xi)$ is the Green's function of the differential expression $(pu')' + qu$ with boundary conditions (3.2). In discussing the last integral we restrict ourselves to the case $q \leqq 0$ in order that $R(u)$ be positive. For each couple of these three integrals it is possible to prove a pair of theorems concerning extrema.

The integrals $D(u)$, $K_0(u)$, $R(u)$ can be approximated as closely as we please by the corresponding integrals in which $u(x)$ possesses an absolutely continuous first derivative. Hence there is no loss of generality in restricting ourselves to the consideration of such functions u.

Theorems I–III concern the orthogonal case and IV the polar case.

THEOREM I. *Among all continuous functions $u(x)$ which give to the integral $D(u)$ a meaning and which are subject to the condition $K_0 = 1$, the boundary conditions* (3.2), *and the infinity of linear conditions*

$$K_i \equiv \int_0^1 kU_i u dx = 0 \quad (i = 1, \cdots, m-1; \quad s+1, \; s+2, \cdots), \tag{3.3}$$

the maximum value λ_s of $D(u)$ is furnished by U_s and the minimum value λ_m is furnished by U_m.

For, in the orthogonal case formulas analogous to (1.20), (1.21) have the simpler form

$$K_0(u) = \sum_1^\infty c_i^2, \quad D(u) = \sum_1^\infty \lambda_i c_i^2$$

and the hypotheses (3.3) reduce the problem to the consideration of relative extrema for quadratic forms in a finite number of variables only,

$$D(u) \equiv \sum_{i=m}^{s} \lambda_i c_i^2 = \text{extremum}, \quad \sum_{i=m}^{s} c_i^2 = 1. \tag{3.4}$$

Since λ_s is the largest of the characteristic numbers here appearing, and λ_m the smallest, the theorem is immediately established.

A consideration of the proof of Theorem I indicates that the reciprocal theorem can be at once deduced.

THEOREM Ia. *Under the boundary and linear conditions of Theorem* I *and* $q(x) \leqq 0$ *the minimum of* $K_0(u)$ *for those values of u which make* $D(u) = 1$ *is* $1/\lambda_s$ *furnished by* U_s *and the maximum is* $1/\lambda_m$ *furnished by* U_m.

THEOREM II. *Among all continuous functions* $u(x)$ *the integral* $R(u)$, *under the conditions* $q \leqq 0$, $K_0 = 1$ *and* (3.2), (3.3), *possesses a maximum value* $1/\lambda_m$ *furnished by* U_m *and a minimum value* $1/\lambda_s$ *furnished by* U_s.

For, on setting $h(x) = k(x)\, u(x)$ the formula (1.19) becomes

$$R(u) = \sum_i \frac{\left(\int_0^1 k(x)u(x)U_i(x)dx\right)^2}{|\lambda_i|}$$

and from (1.20) and the hypotheses, this may be written

$$R(u) = \sum_m^s \frac{c_i^2}{\lambda_i}\cdot \tag{3.5}$$

This with the second formula of (3.4), valid here also, is sufficient to establish the theorem.

THEOREM IIa. *Under the boundary and linear conditions of Theorem* II *the maximum of* K_0 *for those values of u which make* $R(u) = 1$ *is* λ_s *furnished by* U_s *and the minimum is* λ_m *furnished by* U_m.

A consideration of the preceding theorems and of (0.20) suggests another theorem which, with its reciprocal, may be readily proved by means of (3.4) and (3.5):

THEOREM III. *Among all functions $u(x)$ which give $D(u)$ a meaning and are subject to the conditions that $R(u)=1$, $q(x)\leqq 0$, and (3.2), (3.3), the integral $D(u)$ possesses a minimum λ_m^2 furnished by U_m and a maximum λ_s^2 furnished by U_s.*

THEOREM IIIa. *Under the boundary and linear conditions of Theorem* III *the maximum of the integral $R(u)$ subject to the condition $D(u)=1$ is $1/\lambda_m^2$ furnished by U_m and the minimum is $1/\lambda_s^2$ furnished by U_s.*

In the polar case the situation allows only one extremum and the reciprocal theorem will have only one.

THEOREM IV. *Among all continuous functions $u(x)$ which give $D(u)$ a meaning and are subject to the conditions $K_0=1$, (3.2) and (3.3), the integral $D(u)$ possesses a minimum λ_m furnished by U_m, while the maximum is infinite.*

For as in Theorem I, by means of (1.20) and (1.21) and the hypothesis, the problem is reduced to relative extrema of quadratic forms with an infinite number of variables

$$(3.6)\quad D(u) = \sum_m^s \lambda_i c_i^2 - \sum_{-\infty}^{-1} \lambda_i c_i^2 = \text{extremum}\ ; \quad K_0 = \sum_m^s c_i^2 - \sum_{-\infty}^{-1} c_i^2 = 1.$$

On multiplication of the second of these by λ_m and subtraction from the first, there results

$$D(u) - \lambda_m = \sum_{m+1}^{s} c_i^2(\lambda_i - \lambda_m) + \sum_{-\infty}^{-1} c_i^2(\lambda_m - \lambda_i),$$

and since all the coefficients of c_i^2 are positive, it is seen that the minimum is given by $c_m=1$, $c_i=0$ $(i=\cdots, -2, -1; m+1, \cdots, s)$. On the other hand for $c_m=2^{1/2}$, $c_{-n}=1$, and the other c's zero the formulas (3.6) give $D(u)$ the value $2\lambda_m-\lambda_{-n}$ and this may be made as great as is desired by taking n large enough.

THEOREM IVa. *Under the boundary and linear conditions of Theorem* IV *and provided $q\leqq 0$, the maximum $1/\lambda_m$ of the integral K_0 for those values of u which make $D(u)=1$ is furnished by U_m.*

For, on multiplication of the second of the expressions

$$K_0 = \sum_m^s c_i^2 - \sum_{-\infty}^{-1} c_i^2 = \text{max.}, \quad D(u) = \sum_m^s \lambda_i c_i^2 - \sum_{-\infty}^{-1} \lambda_i c_i^2 = 1$$

by $1/\lambda_m$ and subtraction from the first, it follows that

$$K_0 - \frac{1}{\lambda_m} = \sum_{m+1}^{s} c_i^2\left(1 - \frac{\lambda_i}{\lambda_m}\right) + \sum_{-\infty}^{-1} c_i^2\left(\frac{\lambda_i}{\lambda_m} - 1\right)$$

and since all the coefficients of c_i^2 are negative the theorem follows at once.

If in Theorem IV, instead of setting K_0 equal to 1 it is equated to -1 and if in (3.3) the U_i are replaced by U_{-i} the minimum is $-\lambda_m$ furnished by U_{-m}. There is a corresponding reciprocal theorem.

4. Generalization of the extremum problem. The Euler equation and its solutions

If we generalize the problem (1.7), (1.8), (2.1) by seeking the minimum or maximum of

$$D(u) = \int_0^1 (pu'^2 - qu^2)dx\,; \qquad p > 0,\ u(0) = u(1) = 0, \tag{4.1}$$

under the quadratic condition

$$K_0 \equiv \int_0^1 ku^2 dx = 1, \tag{4.2}$$

and the *infinite* number of linear conditions

$$K_i \equiv \int_0^1 kU_i u dx = 0 \qquad (i = 1, \cdots, m-1\,;\ s+1,\ s+2, \cdots\,; (s \geqq m)) \tag{4.3}$$

the Lagrange method suggests the consideration of the absolute minimum of the integral

$$\int_0^1 \Big[pu'^2 - qu^2 + \lambda(v_0' - ku^2) + \sum_1^{m-1} 2\mu_i(v_i' - kU_i u) + \sum_{s+1}^{\infty} 2\mu_i(v_i' - kU_i u)\Big]dx, \tag{4.4}$$

where after the analogy of (1.9) and (1.12) for the finite case, the v's are defined as follows:

$$v_0 = \int_0^x ku^2 dx, \quad v_i = \int_0^x kU_i u dx \quad (i = 1, \cdots, m-1\,;\ s+1, \cdots), \tag{4.5}$$

which may also be written

$$\begin{aligned} &v_0' - ku^2 = 0, \quad v_0(0) = 0, \quad v_0(1) = 1\,; \\ &v_i' - kU_i u = 0, \quad v_i(0) = v_i(1) = 0. \end{aligned} \tag{4.6}$$

It is natural to expect that *the Euler equation will have a form*

$$(4.7)\qquad (pu')' + qu + \lambda ku + \sum_{1}^{m-1}\mu_i kU_i + \sum_{s+1}^{\infty}\mu_i kU_i = 0$$

generalized from (2.2) *and the solutions*

$$(4.8)\qquad u = \alpha u_1(x,\lambda) - \sum_{1}^{m-1}\frac{\mu_i U_i}{\lambda - \lambda_i} - \sum_{s+1}^{\infty}\frac{\mu_i U_i}{\lambda - \lambda_i}$$

will be a generalized form of (2.3).

That the solutions (4.8) actually satisfy the equation (4.7) may be proved by direct substitution. To indicate the line of argument for deriving the Euler equation (4.7), we proceed formally and assume that $u_1(x)$ gives an extremum and set up admissible variations after the usual method. If the fundamental set of functions on which the variations are to be linearly dependent are chosen at random, the number of them must ordinarily be infinite. For, the family

$$(4.9)\qquad Y(x,\epsilon_1,\epsilon_2,\cdots) = u_1 + \sum \epsilon_j\eta_j(k),\qquad \eta_j(0) = \eta_j(1) = 0$$

is subject to a quadratic and an infinity of linear conditions and the ϵ's must be chosen to satisfy them. If, however, it be noted that the linear conditions (4.3) are satisfied by any one of the functions* $U_m,\cdots,U_s$, or any linear combination of them, the problem is reduced to a much simpler one. For example,

$$Y(x,\epsilon) = (1-\epsilon)U_s + (2\epsilon - \epsilon^2)^{1/2}U_l,\qquad m < l < s,$$

satisfies not only the linear conditions but also the quadratic (4.2).

In the general case it is easily seen that the set (4.9) must satisfy the relations

$$K_0 = \int_0^1 k[u_1 + \sum_j \epsilon_j\eta_j(x)]^2 dx = 1,\qquad K_i = \int_0^1 kU_i\sum_j \epsilon_j\eta_j dx = 0,$$

and give to

$$D(\epsilon_1,\epsilon_2,\cdots) = \int_0^1 [p(u_1 + \sum\epsilon_j\eta_j(x))'^2 - q(u_1 + \sum\epsilon_j\eta_j(x)^2]\} dx$$

an extremal value for $\epsilon_i = 0$. It is then necessary that, for the values $\epsilon_1 = \epsilon_2 = \cdots = 0$,

* In the polar case these functions are $\cdots U_{-2}, U_{-1}, U_m, \cdots, U_s$. The argument of this section is in general valid for that case also.

$$\sum_1^\infty \frac{\partial D}{\partial \epsilon_j} d\epsilon_j = 0, \quad \sum_1^\infty \frac{\partial K_0}{\partial \epsilon_j} d\epsilon_j = 0, \quad \sum_1^\infty \frac{\partial K_i}{\partial \epsilon_j} d\epsilon_j = 0$$

$$(i = 1, \cdots, m-1, s+1, s+2, \cdots);$$

hence whatever the multipliers λ, μ_i may be, it follows that

$$d\epsilon_1\left(\frac{\partial M}{\partial \epsilon_1}\right) + \sum_2^\infty d\epsilon_j\left(\frac{\partial M}{\partial \epsilon_j}\right) = 0 \tag{4.10}$$

where

$$M = D - \lambda K_0 - \sum_1^{m-1} 2\mu_i K_i - \sum_{s+1}^\infty 2\mu_i K_i.$$

Let λ, μ_i be determined by the equations

$$\frac{\partial M}{\partial \epsilon_j} = \frac{\partial D}{\partial \epsilon_j} - \lambda\left(\frac{\partial K_0}{\partial \epsilon_j}\right) - \sum_1 \mu_i\left(\frac{\partial K_0}{\partial \epsilon_j}\right) = 0 \qquad (j = 2, 3, \cdots) \tag{4.11}$$

which is possible provided the determinant of the coefficients of λ and μ_i

$$\begin{vmatrix} \int_0^1 k u_1 \eta_2 dx & \int_0^1 k U_1 \eta_2 dx & \cdots & \int_0^1 k U_{m-1} \eta_2 dx & \int_0^1 k U_{s+1} \eta_2 dx & \cdots \\ \int_0^1 k u_1 \eta_3 dx & \int_0^1 k U_1 \eta_3 dx & \cdots & \int_0^1 k U_{m-1} \eta_3 dx & \int_0^1 k U_{s+1} \eta_3 dx & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

is different from zero. This may be ensured by proper choice of the η's; for example, diagonal terms may be made unity and all the other terms zero.

The values of λ, μ_i so chosen are independent of η_1. Hence from the formula $\partial M/\partial \epsilon_1 = 0$ derived by subtracting the infinite set of equations (4.11) from (4.10), the Euler equation (4.7) may be at once derived in the usual way.

Let us return to a discussion of the solutions (4.8) which may be regarded as an infinity-parameter set of plane extremals through the origin. Since the U_i vanish at $x=1$, in order that u vanish at that point also, it is necessary that $u_1(1, \lambda) = 0$. For the minimizing or maximizing extremal of the family it may be shown that $\mu_i = 0$ by the method used in deriving (1.15); in other words the extremum is a solution of the homogeneous equation (1.1). The function u_1 is then a solution of the homogeneous system (1.1), (1.2) and is orthogonal to U_i unless it is a multiple of it. So far as we ascertain from the Euler equation, any one of the functions $U_m, \cdots, U_s$ corresponding to the characteristic numbers $\lambda_m, \cdots, \lambda_s$ might serve as a solution. One of these must give the minimum and one of them the maximum.

To round out the discussion and prepare for the treatment of the Jacobi condition, the problem of extremals may be interpreted in infinity-dimension space $x\,u\,v_0\,v_i$. The Euler equations would in that case consist of (4.7), with the boundary conditions $u(0)=u(1)=0$, together with (4.6); the solutions constituting the infinity-parameter family of extremals through the origin would have a form generalized from (2.4) and would consist of (4.5) and (4.8).

In dealing with this family of extremals passing through the origin, it is natural to consider only those functions for which $\int_0^1 ku^2dx$ is finite; an application of this condition to (4.8) shows that this limitation is equivalent to supposing that $\sum[\mu_i/(\lambda-\lambda_i)]^2$ is limited.

5. The second variation

Despite the introduction of newer methods for the simple problem without auxiliary conditions, the method of second variation still remains standard for isoperimetric problems. It is then natural after the discussion of the Euler equation to proceed to the discussion of δ^2D. For the admissible variations $\eta=\sum\epsilon_j\eta_j$ set up in (4.9) it is a necessary condition that, according as a minimum or maximum is sought, $\delta^2D\geqq 0$ or $\delta^2D\leqq 0$. Since by the nature of the hypotheses, the second variations δ^2K_i are 0, this may also be written

$$\delta^2D+\delta^2K_0+\sum_1^{m-1}\delta^2K_i+\sum_{s+1}^{\infty}\delta^2K_i\geqq 0 \text{ or } \leqq 0.$$

On calculation from (4.4) it is found that

$$\delta^2D=\epsilon^2\int_0^1(p\eta'^2-q\eta^2-\bar{\lambda}k\eta^2)dx \tag{5.1}$$

where $\bar{\lambda}$ is the characteristic number of the extremum solution and where η is subject to the conditions

$$\int_0^1 ku\eta dx=0,\quad \int_0^1 kU_i\eta dx=0\quad (i=1,\cdots,m-1;\ s+1,\cdots). \tag{5.2}$$

By integrating (5.1) by parts and adding multiples of the linear terms (5.2) the second variation may be written

$$\begin{aligned}\delta^2D=-\epsilon^2\int_0^1\eta\Big[(p\eta')'+q\eta+\lambda k\eta+\alpha\delta\lambda k\,u+\sum_1^{m-1}\mu_ikU_i\\ +\sum_{s+1}^{\infty}\mu_ikU_i\Big]dx.\end{aligned} \tag{5.3}$$

In the orthogonal case for the problem* (4.1), (4.2), (4.3) *the second variation related to the minimizing function U_m is positive and that related to the maximizing function U_s is negative.*

For, taking up first the problem of a maximum it is necessary that the integral

$$\int_0^1 (p\eta'^2 - q\eta^2 - \lambda_s k\eta^2)dx \tag{5.4}$$

be negative for all $\eta \neq 0$ satisfying the continuity and boundary conditions and the linear conditions (5.2); it will also satisfy a quadratic condition such as

$$\int_0^1 k\eta^2 dx = c \neq 0\,. \tag{5.5}$$

The problem may be regarded as that of finding a maximum zero of (5.4) for those functions $\eta(x) \neq 0$ which satisfy (5.2) and (5.5). As Bliss has pointed out in similar problems, the original problem for the integral $D(u)$ may itself be put into precisely this form and the results there obtained applied here. The admissible variation η must be linearly dependent on $U_m, \cdots, U_s$; that is, $\eta = \sum_{i=m}^{s} a_i U_i$; on calculation it turns out in a manner analogous to Theorem I of §3 that

$$\delta^2 D = \epsilon^2 \sum_m^s (\lambda_i - \lambda_s)a_i^2 \leqq 0.$$

That $\delta^2 D$ is actually negative may be seen by noting that it could be zero only if $a_m = \cdots = a_{s-1} = 0$; since $\sum a_i^2 \neq 0$, it follows that $a_s \neq 0$. But U_s cannot be an admissible variation for U_s itself since the value of K_0 would be affected; hence $\delta^2 D < 0$.

A similar argument shows that in the problem of a minimum the admissible variations of the functions U_m make $\delta^2 D$ positive.

From analogy with the Legendre condition for the finite problem, we would expect, in order that a maximum exist, that $H_{y'y'} = 2p$ (where H is the integrand of (4.4)) must be negative while for a minimum this same function must be positive. But here we have found a maximum for $p > 0$ in striking contradiction to the theorems for the finite problem. It is evident that there must be some underlying reason why one of these conditions and not the other is satisfied. As will be evident later, an investigation of

* For the minimum problem in the polar case (Theorem 4, § 3), it follows in similar fashion that

$$\delta^2 D = \epsilon^2 \sum_m^s a_i^2(\lambda_i - \lambda_m) + \epsilon^2 \sum_{-\infty}^{-1} a_i^2(\lambda_m - \lambda_i) > 0.$$

the Jacobi condition for the problem is fundamental before any appeal can be made to the Legendre condition.

It may be noted that all the admissible variations of the maximum problem are contained in the family of extremals of the minimum problem, while a part only of the admissible variations for the minimum problem are contained among the extremals of the maximum problem.

6. The Jacobi condition for the infinite problem

In (1.3) there was set up a two-parameter family $\alpha u_1(x, \lambda)$ of solutions of the homogeneous equation (1.1) and in (4.8) an infinity-parameter family

$$Y(x,\lambda,\mu_i) \equiv \alpha u_1(x,\lambda) - \sum_{1}^{m-1} \mu_i \frac{U_i}{\lambda - \lambda_i} - \sum_{s+1}^{\infty} \mu_i \frac{U_i}{\lambda - \lambda_i} \tag{6.1}$$

of plane extremals which are solutions of the Euler equation (4.7) and which pass through the origin, the parameters μ_i being restricted to those values which make $\sum[\mu_i/(\lambda-\lambda_i)]^2$ finite. By means of the auxiliary variables $v_0(x), v_1(x), \cdots, v_{m-1}(x); v_{s+1}(x), \cdots$ as defined in (4.6), extremals were also set up in space of infinity dimensions xuv_0v_i. To every extremal (6.1) of the xu space corresponds an extremal in the higher space. Among the questions which present themselves is that concerning the existence of a field in the neighborhood of the minimizing extremal in infinity dimensions. Does there exist a region about this curve through each point of which there passes a unique extremal of the family in infinity-dimensional space? In other words, do there exist constants α, λ, μ_i such that for these values an extremal (6.1), (4.5) passes through the origin and any other designated point? Is there a one-to-one correspondence between the xuv_0v_i space and the $\alpha\lambda\mu_i$ space? Or, on the contrary, is one extremal cut by a neighboring one before the end of the interval 0, 1 is reached: that is, is the point conjugate to $x=0$ in the extended sense within the interval? The condition for a conjugate point has been developed in §2 at considerable length for the finite problem and it is not necessary in extending it formally to the infinite problem that great detail be given.

For a conjugate point an infinity of conditions corresponding to (2.5) must be satisfied:

$$\alpha\delta\lambda\frac{\partial u_1}{\partial\lambda} + \delta\alpha u_1 - \sum \delta\mu_i \frac{U_i}{\lambda - \lambda_i} = 0,$$

$$2\alpha^2\delta\lambda \int_0^x k u_1 \frac{\partial u_1}{\partial\lambda} dx + 2\alpha\delta\alpha \int_0^x k u_1^2\, dx - 2\alpha \sum \frac{\delta\mu_i}{\lambda - \lambda_i} u_1 U_i dx = 0, \tag{6.2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

This leads to a consideration of the infinite determinant:

(6.3)

$$\begin{vmatrix} \dfrac{\partial u_1}{\partial\lambda} & u_1 & \cdots & U_{m-1} & U_{s+1} & \cdots \\ \displaystyle\int_0^x ku_1\frac{\partial u_1}{\partial\lambda}dx & \displaystyle\int_0^x ku_1^2\,dx & \cdots & \displaystyle\int_0^x kU_{m-1}u_1dx & \displaystyle\int_0^x kU_{s+1}u_1dx & \cdots \\ \displaystyle\int_0^x kU_1\frac{\partial u_1}{\partial\lambda}dx & \displaystyle\int_0^x kU_1u_1dx & \cdots & \displaystyle\int_0^x kU_1U_{m-1}dx & \displaystyle\int_0^x kU_{s+1}U_1dx & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

Denoting by $F_{m+1}(x)$, $F_{m+2}(x)$, $\cdots$ the principal $(m+1)$th, $(m+2)$th, $\cdots$ order minors in its upper left-hand corner, it may easily be shown that the determinant (6.3), regarded as the limit of these minors, is a bounded function of x, being 0 at $x=0$ and 1 at $x=1$. From the previous paper* we have the theorems that each of the functions $F_{m+1}(x)$, $F_{m+2}(x)$, $\cdots$ are 0 at $x=0$ and positive elsewhere, being 1 at $x=1$, and that the quotient F_{m+p}/F_m is, for all p, a monotone function ranging from 0 at $x=0$ to 1 at $x=1$. A passage to the limit gives a bounded function (6.3).

The formal analogon of the Jacobi condition may then be stated as follows: *In order that there be no conjugate point in the interval, the infinite determinant* (6.3) *must have no zero other than* $x=0$ *in the interval.*

To give a formal indication of the necessity of this condition let us assume that the equations (6.2) are satisfied for a point x_1 within the interval and prove that this involves the vanishing of δ^2D. We have seen (5.3) that the second variation may be written

$$(6.4)\qquad \delta^2D = -\epsilon^2\int_0^1 \eta\Big[(p\eta')' + q\eta + \lambda_m k\eta + \alpha\delta\lambda ku + \sum \mu_i kU_i\Big]dx.$$

Since for the minimizing extremal $\lambda=\lambda_m$ and $\mu_i=0$, it follows that $\mu_i=\delta\mu_i$, $\lambda-\lambda_m=\delta\lambda$; the expression in brackets in the integrand of (6.4) has then the form

$$(6.5)\qquad (p\eta')' + q\eta + \lambda_m k\eta + \alpha\delta\lambda ku + \sum\delta\mu_i kU_i.$$

It is readily shown that the substitution of the expression on the left of the first equation of (6.2) will make (6.5) zero. If then in the interval $0, x_1$ we

* Loc. cit., Mathematische Annalen, vol. 68, p. 289.

choose for the variation η this expression in (6.2) and in the sub-interval x_1, 1 set $\eta=0$, the second variation $\delta^2 D$ vanishes.

That $\delta^2 D$ can actually in that case be made negative can be shown by the following argument. Referring to the discussion of the second variation in §5 it may be noted that were η to furnish the minimum for the integral (5.4) under the conditions (5.2), (5.5), thus making the Euler equation of this subsidiary problem the same as that of the original extremum problem, the solution would have all its derivatives continuous at x_1, which is obviously not the case here. Hence the variation η chosen above does not give a minimum to the second variation and $\delta^2 D$ can be made negative. This would indicate that for a minimum the point conjugate to $x=0$ cannot be within the interval; and it indicates also that there must be a conjugate point in the interval if there is to be a maximum.

To consider the relation between the infinite determinants (6.3) for various values of m and s let us denote by D_{14} the infinite determinant obtained by setting $m=1$ and $s=3$ and by D_{134} that obtained by setting $m=1$, $s=2$. The latter contains one more row and column than the former and as is indicated in the Lemma in §2, we have the formula

$$\frac{d}{dx}\frac{D_{14}}{D_{134}} = \frac{\lambda_2-\lambda_3}{pD_{134}^2}\alpha_{14}^2,$$

where α_{14} is a certain first minor of D_{134}. In other words the discussion parallels exactly that of §2 except that instead of a finite number of terms there is an infinite number. Each of the infinite determinants obtained by dropping out any finite number of columns and the corresponding rows (taken in order or scattered here and there throughout the determinant) can have no zero within the interval. By dropping out any column and corresponding row the zero of the determinant moves to the left. Since the ratio of any determinant to that of order lower by one is monotone in the interval 0, 1 the same will be true concerning the ratio of any two in the scale provided the one is contained in the other.

7. Hamilton function. Hilbert integral. Weierstrass condition

Assuming that the Jacobi condition is satisfied in the interval 0, 1, consider a point in the infinity-dimensional field about the maximizing or minimizing extremal. Through the origin and this point whose abscissa is x there will be an extremal of the family

$$u = \alpha u_1(x,\lambda) - \sum_{1}^{m-1}\frac{\mu_1 U_i}{\lambda-\lambda_i} - \sum_{s+1}^{\infty}\frac{\mu_i U_i}{\lambda-\lambda_i}, \tag{7.1}$$

$$v_0 = \int_0^x ku^2 dx, \quad v_i = \int_0^x kU_i u dx. \tag{7.2}$$

The Hamilton function is defined to be the integral

$$W(x,u,v_0, v_1, \cdots, v_{m-1}, v_{s+1}, \cdots) = \int_0^x (pu'^2 - qu^2) dx \tag{7.3}$$

taken along this extremal. Since along this curve the relations (7.2) are satisfied, the integral may also be written

$$W = \int_0^x [pu'^2 - qu^2 + \lambda(v_0' - ku^2) + \sum_1^{m-1} \mu_i(v_i' - kuU_i)$$
$$+ \sum_{s+1}^{\infty} \mu_i(v_i' - kuU_i)] dx.$$

By the method usual in such cases* the derivatives may be calculated formally as follows:

$$\frac{\partial W}{\partial x} = p\wp^2 - qu^2 - \wp(2p\wp) - \sum_1^{m-1} \mu_i \wp_i - \sum_{s+1}^{\infty} \mu_i \wp_i ;$$
$$\frac{\partial W}{\partial u} = 2p\wp ; \quad \frac{\partial W}{\partial v_0} = 0 ; \quad \frac{\partial W}{\partial v_i} = \mu_i,$$

where $\wp$ is the slope of the projection on the xu plane of the space extremal through the given point and $\wp_i$ the slope of the projection of the space extremal on the xv_i plane. Because of the linear character of the conditions, it follows from the definitions that both $\wp_i$ and v_i' are equal to the value of kuU_i at the point in question and hence are equal to one another. The differential dW may then be written

$$dW = \left[- p\wp^2 - qu^2 - \sum_1^{m-1} \mu_i v_i' - \sum_{s+1}^{\infty} \mu_i v_i' \right] dx + 2p\wp du$$
$$+ \sum_1^{m-1} \mu_i dv_i + \sum_{s+1}^{\infty} \mu_i dv_i = - (p\wp^2 + qu^2) dx + 2p\wp du.$$

Because this is a perfect differential its integral

$$\int_0^x (- p\wp^2 - qu^2 + 2p\wp u') dx \tag{7.4}$$

is independent of the path and is the analogon of the Hilbert independent integral for this problem.

* Bolza, p. 599.

To set up the Weierstrass formula let us compare the value of $D(u)$ taken for the interval 0, 1 along a curve C of admissible variation, which must satisfy the equations (7.2), with its value along the minimizing extremal. The integral (7.4) taken throughout the interval along the minimizing extremal is the Hamilton function and represents the minimum. Its value along the admissible variation is the same. Hence

$$\Delta J = \int_C [(pu'^2 - qu^2) + (p\varphi^2 + qu^2 - 2p\varphi u')]dx$$

and on setting $E(x, u, u', \varphi) = p(u' - \varphi)^2$ this may be written

$$\Delta J = \int_C E(x, u, u', \varphi)dx.$$

The conditions $E(x, u, u', \varphi) \geqq 0$, $E(x, u, u', \varphi) \leqq 0$ *would be the analogons of the Weierstrass conditions for minimum and maximum respectively in the finite problem.*

Here $E(x, u, u', \varphi) \geqq 0$ for both minimum and maximum, and the significance of this condition has entirely disappeared.

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A CONTRIBUTION TO THE THEORY OF FUNDAMENTAL TRANSFORMATIONS OF SURFACES*

BY

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Introduction

Two surfaces are said to be related to one another by a fundamental transformation, that is, by a transformation F, if the developables of the congruence of lines joining corresponding points on the surfaces cut the surfaces in conjugate nets of curves. It is assumed that neither of these nets is a focal net of the congruence. The nets on the surfaces are also said to correspond by the transformation F.

Although many well known transformations of surfaces are special types of transformations F, the general case was treated in detail but recently, by Eisenhart† and Jonas.‡ In a recent paper Graustein§ introduced into the study of these transformations a projective invariant which was the generalization of the invariant of a parallel map.|| Certain important theorems concerning this invariant were obtained whose nature indicates that transformations F can be investigated to advantage by means of it. We call this invariant the invariant C.

When studied in terms of tangential coördinates, transformations F present a complete duality among the elements involved. In this way, a second invariant, the invariant H, is obtained which is dual to the invariant C. The invariant C is equal to the cross ratio in which a pair of corresponding points of the surfaces in the relation F is divided by the focal points of the line joining them. Dually, the invariant H is equal to the cross ratio in which a pair of corresponding tangent planes to the two surfaces is divided by the focal planes through their line of intersection.

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† Cf. Eisenhart's treatise, *Transformations of Surfaces*, Princeton, 1923, which deals primarily with these transformations. We shall follow the notation employed in this book, and shall refer to it as Eisenhart, T. S.

‡ Jonas, Sitzungsberichte, Berliner Mathematische Gesellschaft, vol. 14 (1915), pp. 103 ff.

§ W. C. Graustein, *An invariant of a general transformation of surfaces*, Bulletin of the American Mathematical Society, vol. 32 (1926), p. 357 ff.

|| W. C. Graustein, *Parallel maps of surfaces*, these Transactions, vol. 23 (1922), pp. 298–332.

It is the purpose of this paper to make a study of transformations F based upon these invariants. In fact, the invariants C and H form a tool by means of which many theorems are found which do not easily lend themselves to proof by the classical methods. Fundamental existence questions which arise concerning the conditions on the invariants C and H and on the nets in the transformation F are readily answered. The relations between the invariants C and H and the surfaces in the transformation also yield interesting consequences.

The invariant C is introduced in Part I, which also contains a fundamental theorem for transformations F of a given net having a given invariant C. The analogous work for the invariant H is done in Part II. The invariants C and H of a transformation F which is the product of two such transformations are also discussed in these two parts.

Transformations F and nets of special type are discussed in Part III. The last part, Part IV, is devoted to the application of some of the results obtained to transformations of Ribaucour.

I. The invariant C of a transformation F

1. **Fundamental equations.** A congruence of lines G and a net N are said to be *conjugate* to one another if the curves of N, which is assumed not a focal net of G, lie on the developables of G. Two nets N and N_1 are then related to one another by a transformation F if the congruence G of lines joining corresponding points of these nets is conjugate to both nets. The congruence G is known as the *conjugate congruence* of the transformation F.

Consider a surface $S: x = x(u, v)$* on which the parametric curves form a net N, which has for its point equation

$$\frac{\partial^2 \theta}{\partial u \partial v} = \frac{\partial \log a}{\partial v} \frac{\partial \theta}{\partial u} + \frac{\partial \log b}{\partial u} \frac{\partial \theta}{\partial v} \cdot \dagger \tag{1.1}$$

To obtain an F transform of N we have first to find a congruence G conjugate to N, and then a net N_1 conjugate to G. A net N', parallel to N, and traced by the point x', where

$$\frac{\partial x'}{\partial u} = h \frac{\partial x}{\partial u}, \qquad \frac{\partial x'}{\partial v} = l \frac{\partial x}{\partial v}, \tag{1.2}$$

determines G in that the point coördinates of N' serve as direction parameters

* I.e., $x^i = x^i(u, v)$, $i = 1, 2, \cdots, n$.

† Eisenhart, T. S., § 2.

for the lines of G. A solution θ of the point equation (1.1) of N will determine a net N_1, conjugate to G, whose point coördinates are

$$x_1 = x - \frac{\theta}{\theta'} x'. \tag{1.3}$$

Here θ' is a solution of the point equation of N' corresponding to θ; i.e., it satisfies the equations

$$\frac{\partial \theta'}{\partial u} = h \frac{\partial \theta}{\partial u}, \qquad \frac{\partial \theta'}{\partial v} = l \frac{\partial \theta}{\partial v}. \tag{1.4}$$

The net N_1 is said to be an F transform of N by means of the solution θ of its point equation and along the congruence G.

The lines of intersection of corresponding tangent planes to the surfaces of N and N_1 also generate a congruence called the ***harmonic congruence*** of the transformation. For a line L of this congruence the focal points F_1, F_2 have the coördinates*

$$F_1 : x - \frac{\theta}{\frac{\partial \theta}{\partial u}} \frac{\partial x}{\partial u}, \qquad F_2 : x - \frac{\theta}{\frac{\partial \theta}{\partial v}} \frac{\partial x}{\partial v}; \tag{1.5}$$

and hence are the intersections of L with the focal planes of the corresponding line of G.

2. **The invariant** C.† A transformation F of the net N into the net N_1 establishes a projective correspondence between the pencils of the tangent lines to the surfaces of these nets at corresponding points x and x_1. These pencils of tangent lines meet the line of intersection L of their planes (the tangent planes to the surfaces of N and N_1 at x and x_1, respectively) in projective ranges of points. In this projectivity the fixed points are the focal points F_1 and F_2. If D and D_1 are a pair of corresponding points of the two ranges on L, the invariant of the projectivity is

$$C = (DD_1, F_1F_2). \tag{2.1}$$

The function C is a projective invariant of the transformation F which we shall call the ***conjugate invariant***, or, briefly, the ***invariant*** C.

The invariant C has another geometric significance.‡ It is the cross ratio

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* Eisenhart, T. S., § 17.

† W. C. Graustein, *An invariant of a general transformation of surfaces*, § 5.

‡ Ibid., § 3.

in which the points x and x_1 of the nets N and N_1 are divided by the focal points z and y of the line of G; i.e.,

$$C = (xx_1, zy). \tag{2.2}$$

For the transformation F discussed in §1, the invariant C is found to be

$$C = \frac{t}{s},\text{*} \tag{2.3}$$

where

$$t = h\theta - \theta', \quad s = l\theta - \theta'. \tag{2.4}$$

Two nets are said to be *radial transforms*† of one another when the lines joining corresponding points are concurrent. We agree to admit radial transformations into the category of transformations F, and point out that $C=1$ is characteristic of them.

Finally, we note that the invariant C of the inverse of a transformation F is equal to the reciprocal of that of the original transformation.

3. **Fundamental theorem.** Equations (2.3), (2.4), (1.1), and (1.2), combined with the condition of compatibility of the equations (1.2), yield the relations

$$\begin{aligned} \frac{\partial \log t}{\partial v} &= \left(1 - \frac{1}{C}\right)\frac{\partial}{\partial v}\log\frac{\theta}{a}, \\ \frac{\partial \log s}{\partial u} &= (1 - C)\frac{\partial}{\partial u}\log\frac{\theta}{b}\cdot \end{aligned} \tag{3.1}$$

We now form the difference between the derivative of the first of these equations with respect to u and that of the second with respect to v and obtain the equation

$$\frac{\partial^2 \log C}{\partial u \partial v} + \frac{\partial}{\partial u}\left[\left(\frac{1}{C} - 1\right)\frac{\partial}{\partial v}\log\frac{\theta}{a}\right] + \frac{\partial}{\partial v}\left[(1 - C)\frac{\partial}{\partial u}\log\frac{\theta}{b}\right] = 0 \tag{3.2}$$

as a condition on the invariant C and the solution θ of the point equation of the net N, for the transformation F.

Suppose now that we have a net N with (1.1) as its point equation, of which θ is a given solution. Given also a function $C(u, v)$ satisfying (3.2). The system of equations (3.1) combined with $sC=t$ is then compatible,

* Ibid., § 5.

† Eisenhart, T. S., § 14.

and by means of it two functions t and s are defined to within the same multiplicative constant. The function

$$\phi = \frac{t - s}{\theta} \tag{3.3}$$

is defined also to within this same multiplicative constant, and is found to satisfy the adjoint equation of (1.1), namely

$$\frac{\partial^2\phi}{\partial u\partial v} + \frac{\partial \log a}{\partial v}\frac{\partial\phi}{\partial u} + \frac{\partial \log b}{\partial u}\frac{\partial\phi}{\partial v} + \frac{\partial^2 \log ab}{\partial u\partial v} = 0. \tag{3.4}$$

Consequently the relation

$$h - l = \phi \tag{3.5}$$

will determine two functions h and l to within a common additive constant of integration, which serve to define a net $N':(x')$ parallel to N, by means of equations similar to (1.2).* The function $\theta' = h\theta - t = l\theta - s$ is then a solution of the point equation of N' corresponding to θ.

Thus we have found transformations F of N by means of the given θ having the given function $C(u, v)$ as their common invariant C. The nets N_1 determined in this manner as F transforms of N are given by the equation

$$x_1 = x - \frac{\theta}{\theta' + n\theta}(x' + nx),\dagger \tag{3.6}$$

in which n is an arbitrary constant.

FUNDAMENTAL THEOREM I. *A solution θ of the point equation* (1.1) *of a net N, and a function $C(u, v)$ which satisfies* (3.2) *determine ∞^1 nets N_1 which are F transforms of N by means of θ having as their invariant C the given function $C(u, v)$. Any two of the nets N_1 are radial transforms of one another.*

The last part of the theorem can be proved directly from (3.6); but it will be made evident by the corollary of §6.

4. **Conjugate triads.** If N_1 and N_2 are F transforms of the net N by essentially different solutions, θ_1 and θ_2 $(\theta_1 \neq c\theta_2)$, of its point equation (1.1), but along the same conjugate congruence, they are themselves in relation F. The transformation F carrying N into N_i $(i=1, 2)$ we indicate by F_i, and that carrying N_1 into N_2 by F_3; then

* Eisenhart, T. S., § 4, (18), (19), (20) and also the next theorem stated there.

† This result is obtained by availing ourselves of a translation of the coördinate axes.

$$F_1F_3 = F_2.$$

Three nets so related to one another will be referred to as a *conjugate triad* of nets.

From (1.5) it is noted that the transformations F_1 and F_2, and therefore also F_3, have different harmonic congruences, corresponding lines of which are concurrent.

If we indicate the invariant C of F_i by C_i $(i=1, 2, 3)$, we have from (2.2)

$$C_1 = (xx_1, zy), \quad C_2 = (xx_2, zy), \quad C_3 = (x_1x_2, zy). \tag{4.1}$$

Hence

$$C_1C_3 = C_2. \tag{4.2}$$

5. **Harmonic triads.** Suppose now that N_1 and N_2 are F transforms of N by means of the same solution θ of its point equation (1.1), but along different conjugate congruences. The nets N_1 and N_2 will be, in this case also, F transforms of one another.* Three nets related to one another in this manner will be referred to as a *harmonic triad* of nets. Using the same symbolism as in the preceding section we may again write

$$F_1F_3 = F_2.$$

Any two of the three nets in a harmonic triad are obtained as F transforms of the third by means of the same solution of its point equation. Because of this fact, we see from (1.5) that the three transformations F involved have the same harmonic congruence. It is to be noted also that the transformations F in a harmonic triad of nets have different conjugate congruences, corresponding lines of which are coplanar.

Using the definition of C as embodied in (2.1), we conclude that here, too,

$$C_1C_3 = C_2.$$

6. **Product of two transformations F.** Suppose that the net N_i is transformed into the net N_j by the transformation F_k $(i, j, k=1, 2, 3$ cyclically), and let L_k be a line of the harmonic congruence of F_k. Since L_k is the intersection of the tangent planes to N_i and N_j, the three lines L must be either concurrent, or all three coincident. If the lines L are concurrent, the triangle formed by the focal points at the intersections of the tangents to the u-curves with one another (cf. (1.5)) will be in the relation

* Eisenhart, T. S., § 20. Eisenhart has applied the term *triad* to what we call a harmonic triad, and has given no name to our conjugate triad. These terms have been introduced in the light of the duality existing between the two types of triads.

of Desargues with that formed by the other three focal points. Each of the sides of one of these triangles intersects the corresponding side of the other triangle in a point of one of the three nets. Hence, corresponding points of the three nets are collinear and the three nets form a conjugate triad.

If corresponding lines L are coincident, the three transformations F have the same harmonic congruence; that is, the three nets form a harmonic triad.*

This result combined with those of §§4, 5, yields

THEOREM II. *If the product of two transformations F is a transformation F, the three nets in question form either a conjugate or a harmonic triad*; *and, in either case, the invariant C of the product transformation is equal to the product of those of the two given transformations.*

As an immediate consequence, we conclude

. COROLLARY. *If two transformations F of a net by the same solution of its point equation have equal invariants C, the two F transforms are radial transforms of one another*; *and conversely, if two non-radial F transforms of a net N along different conjugate congruences are radial transforms of one another, the two transformations F are by means of the same solution of the point equation of N and have equal invariants C.*

7. **Transformations F in homogeneous point coördinates.** The point equation of a net N on a surface $S: x = x(u, v)$ in a space referred to a homogeneous point coördinate system is of the form

$$\frac{\partial^2\theta}{\partial u \partial v} = \frac{\partial \log a}{\partial v}\frac{\partial\theta}{\partial u} + \frac{\partial \log b}{\partial u}\frac{\partial\theta}{\partial v} + c\theta.\dagger \tag{7.1}$$

If θ is a solution of this equation, the point x_1 defined by

$$\frac{\partial x_1}{\partial u} = t\frac{\partial}{\partial u}\left(\frac{x}{\theta}\right), \quad \frac{\partial x_1}{\partial v} = s\frac{\partial}{\partial v}\left(\frac{x}{\theta}\right) \tag{7.2}$$

traces a net N_1 which is an F transform of N. The points with the coördinates $\partial x_1/\partial u$ and $\partial x_1/\partial v$ are the focal points F_1 and F_2 (cf. (1.5)) of the line of the

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* There is also the case in which the three nets N form both a conjugate and a harmonic triad. Two nets which, with the net N of §1, form such a configuration are those along the same congruence conjugate to N and by means of the same solution θ of the point equation of N. However, the two "corresponding" solutions of the point equation of N' differ by a constant (θ' and θ'+ constant (cf. (1.4)). Three nets so related may be considered as forming either type of triad.

† Eisenhart, T. S., §§ 30, 37, 38.

harmonic congruence. The invariant C of the transformation F is found to be

$$C = \frac{t}{s} \cdot \tag{7.3}$$

The condition of compatibility of (7.2) assumes the form (3.1) by virtue of the fact that θ is a solution of (7.1).

For a radial transformation $C=1$; i.e., $t=s$. In this case, because of (3.1), both t and s are equal to the same constant. Equations (7.2) can then be integrated and

$$x_1 = \frac{x}{\theta} + \rho \tag{7.4}$$

is obtained as the equation of a radial transformation. Here ρ represents the coördinates of the center of the transformation.

From (3.1) we deduce that the condition on the invariant C in this case is precisely of the same form as (3.2).

Given, conversely, a net N whose point equation is (7.1), a solution θ of (7.1), and a function $C(u, v)$ satisfying the condition (3.2). Just as in §3, we find that a net N_1 is determined as an F transform of N to within a radial transformation*; and that the invariants C of the transformations F are equal to the given function C.

If we replace the coördinates x of N by θx, where θ is a solution of (7.1), the point equation assumes a similar form but with $c=0$.† In this event, equations (7.2) become similar in form to those for a parallel map in terms of non-homogeneous coördinates. In fact, as Eisenhart‡ points out, the study of transformations F in terms of homogeneous coördinates can be made in this way analytically equivalent to that of parallel maps in terms of non-homogeneous coördinates. In such a development the invariant C of the transformation F corresponds to the invariant of the parallel map.§

II. The invariant H of a transformation F

8. Nets and transformations F in terms of tangential coordinates. The tangential coördinates of a surface S whose point coördinates are $x=x(u, v)$|| are the direction cosines of its normal:

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* Inasmuch as we cannot avail ourselves of a translation as we did in § 3, it cannot be concluded here that there are only ∞^1 transformations F determined.

† Eisenhart, T. S., § 37.

‡ Ibid., p. 89.

§ W. C. Graustein, *Parallel maps of surfaces.*

|| From now on we restrict ourselves to three-dimensional space.

$$(8.1)\qquad \zeta = \frac{1}{D}\frac{\partial x}{\partial u}\times\frac{\partial x}{\partial v},\quad (\zeta\,|\,\zeta) \equiv 1,^*$$

and the distance of the tangent plane from the origin:

$$(8.2)\qquad \omega = (\zeta\,|\,x).$$

The parametric curves of S will form a conjugate net N if and only if the tangential coördinates ζ and ω satisfy an equation of the form

$$(8.3)\qquad \frac{\partial^2\lambda}{\partial u\partial v} = \frac{\partial\log\alpha}{\partial v}\frac{\partial\lambda}{\partial u} + \frac{\partial\log\beta}{\partial u}\frac{\partial\lambda}{\partial v} + \gamma\lambda,\dagger$$

which is known as the *tangential equation* of N.

Let (8.3) be the tangential equation of the net N of §1; and let ζ, ω be its tangential coördinates. The tangential coördinates of N_1 will be written ζ_1, ω_1. The function $\lambda=(\zeta|x')$ is the fourth tangential coördinate of N', and is also a solution of (8.3).

The net N_1' traced by the point

$$(8.4)\qquad x_1' = \frac{x'}{\theta'}$$

is a radial transform of N' by means of θ', and is parallel to N_1.‡ Its fourth tangential coördinate is $\lambda_1=(\zeta_1|x_1')$.

The transformation F of §1 is represented in terms of tangential coördinates by the equations

$$(8.5)\qquad \begin{aligned} \frac{\partial}{\partial u}\left(\frac{\zeta_1}{\lambda_1}\right) &= \bar{t}\frac{\partial}{\partial u}\left(\frac{\zeta}{\lambda}\right), & \frac{\partial}{\partial v}\left(\frac{\zeta_1}{\lambda_1}\right) &= \bar{s}\frac{\partial}{\partial v}\left(\frac{\zeta}{\lambda}\right),\\ \frac{\partial}{\partial u}\left(\frac{\omega_1}{\lambda_1}\right) &= \bar{t}\frac{\partial}{\partial u}\left(\frac{\omega}{\lambda}\right), & \frac{\partial}{\partial v}\left(\frac{\omega_1}{\lambda_1}\right) &= \bar{s}\frac{\partial}{\partial v}\left(\frac{\omega}{\lambda}\right).\S \end{aligned}$$

When we reconcile these equations with those of §1, we find that

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* The inner product of the two triples x: (x^1, x^2, x^3), y: (y^1, y^2, y^3) is represented by $(x|y)$; and their outer product by $x\times y$. In this way we have $(x\times y|z)=(xyz)$, the latter term being the determinant of x, y and z. Also

$$D^2 = EG-F^2 = \left(\frac{\partial x}{\partial u}\times\frac{\partial x}{\partial v}\,\middle|\,\frac{\partial x}{\partial u}\times\frac{\partial x}{\partial v}\right),$$

where E, F, G are the coefficients of the square of the linear element of S.

† Eisenhart, *Differential Geometry*, §§ 66, 67.

‡ Eisenhart, T. S., § 15.

§ Eisenhart, T. S., § 52, (26).

$$\bar{t} = -\frac{e_1 D_1 \theta'^4}{eDt^2 s}, \quad \bar{s} = -\frac{g_1 D_1 \theta'^4}{gDts^2} . * \tag{8.6}$$

9. **The invariant H.** At the focal point F_1 (cf. (1.5)) of the line L of the harmonic congruence, the tangent plane to any ruled surface of that congruence is the focal plane there; and a similar situation exists at the other focal point F_2. Consider an arbitrary ruled surface of the harmonic congruence. The coördinates of the points of contact, P and P_1, of the tangent planes to N and N_1 with this arbitrary ruled surface depend linearly on the value of du/dv for this ruled surface. Thus, as du/dv is allowed to vary, there are defined on the line of the harmonic congruence two ranges of points, the one traced by P and the other by P_1, which are in projective correspondence. The fixed points of this projectivity are F_1 and F_2. Accordingly, the invariant of this projectivity

$$H = (PP_1, F_2F_1) \tag{9.1}$$

is a projective invariant of the transformation F which we call the ***harmonic invariant***, or briefly, the ***invariant H***.

Evidently the invariant H of the inverse of a transformation F is the reciprocal of that of the given transformation.

The value of H may be obtained by computing the coördinates of P and P_1 in terms of point coördinates using the formulas of §1. It is found that

$$H = \frac{e_1 g}{g_1 e} \cdot \frac{1}{C} \cdot \tag{9.2}$$

An alternative method of obtaining this result is to use the fact that the cross ratio (9.1) is equal to that in which the tangent planes to N and N_1 are divided by the focal planes at F_1 and F_2. The tangential coördinates of the first two planes mentioned are respectively ζ, ω and ζ_1, ω_1. Those of the focal planes are *proportional* to

$$\frac{\zeta_1}{\lambda_1} - \bar{t}\frac{\zeta}{\lambda}, \quad \frac{\omega_1}{\lambda_1} - \bar{t}\frac{\omega}{\lambda} \quad \text{and} \quad \frac{\zeta_1}{\lambda_1} - \bar{s}\frac{\zeta}{\lambda}, \quad \frac{\omega_1}{\lambda_1} - \bar{s}\frac{\omega}{\lambda} \cdot \dagger$$

Hence (9.1) becomes

$$H = \frac{\bar{t}}{\bar{s}} \cdot \tag{9.3}$$

By virtue of (8.6) this result is seen to be equivalent to (9.2).

* Here e, f, g are the coefficients of the second fundamental quadratic form for the surface of the net N.

† Eisenhart, T. S. § 52, (27).

A somewhat different geometric consideration brings to light another meaning of the invariant H. As we leave the point x of N in the direction du/dv, the tangent plane there twists about the conjugate direction. Consequently we can obtain the invariant H just as the invariant C was obtained,* except that each direction is now to be replaced by its conjugate direction. In this way, F_1, F_2 of (2.1) are interchanged; and D and D_1 are replaced by P and P_1, where the latter two points are the intersections of the line of the harmonic congruence with those tangent lines at x and x_1 to the surfaces of N and N_1 in the directions conjugate to the lines from x and x_1 to D and D_1.

10. **Perspective transformations.** The conditions of integrability of (8.5) can be written in the form

$$\frac{\partial \log \bar{t}}{\partial v} = \left(1 - \frac{1}{H}\right)\frac{\partial}{\partial v} \log \frac{\lambda}{\alpha}, \tag{10.1}$$

$$\frac{\partial \log \bar{s}}{\partial u} = (1 - H)\frac{\partial}{\partial u} \log \frac{\lambda}{\beta},$$

where we have made use of (8.3) and (9.3).

If $H=1$, $\bar{t}=\bar{s}$ and their common value is seen from equations (10.1) to be constant. Equation (8.5) can be integrated in this case and we obtain

$$\zeta_1 = \frac{\zeta + \lambda\rho}{\Lambda}, \quad \omega_1 = \frac{\omega + \lambda r}{\Lambda}; \quad \lambda_1 = \frac{\lambda}{\Lambda}, \tag{10.2}$$

where

$$\Lambda = [(\zeta + \lambda\rho) \mid (\zeta + \lambda\rho)]^{1/2}. \tag{10.3}$$

Hence the lines of intersections of corresponding tangent planes to N and N_1 all lie in a fixed plane. We refer to a transformation of this type as a ***perspective transformation.***

Inasmuch as a perspective transformation in terms of tangential coördinates is analytically equivalent to a radial transformation in terms of point coördinates, we may say that they are duals of one another.

A simple example of a perspective transformation is the parallel map, in which the plane of perspectivity is the plane at infinity.

11. **Fundamental theorem.** From equations (10.1) in a manner similar to that of §3 we obtain, as the condition on the invariant H of the transformation (8.5),

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* Cf. § 2, above.

$$(11.1)\qquad \frac{\partial^2 \log H}{\partial u \partial v}+\frac{\partial}{\partial u}\left[\left(\frac{1}{H}-1\right)\frac{\partial}{\partial v}\log\frac{\lambda}{\alpha}\right]+\frac{\partial}{\partial v}\left[(1-H)\frac{\partial}{\partial u}\log\frac{\lambda}{\beta}\right]=0\,.$$

Conversely, given a net N whose tangential equation is (8.3), a solution λ of (8.3), and a function $H(u, v)$ satisfying (11.1). Equations (10.1) and $\bar{s}H=\bar{t}$ will then form a compatible system by means of which two functions $\bar{t}$ and $\bar{s}$ are defined to within the same multiplicative constant. The equations

$$(11.2)\qquad \begin{aligned} &\frac{\partial w_i}{\partial u}=\bar{t}\frac{\partial}{\partial u}\left(\frac{\zeta^i}{\lambda}\right), \qquad \frac{\partial w_i}{\partial v}=\bar{s}\frac{\partial}{\partial v}\left(\frac{\zeta^i}{\lambda}\right)^* \qquad (i=1,2,3),\\ &\frac{\partial w_4}{\partial u}=\bar{t}\frac{\partial}{\partial u}\left(\frac{\omega}{\lambda}\right), \qquad \frac{\partial w_4}{\partial v}=\bar{s}\frac{\partial}{\partial v}\left(\frac{\omega}{\lambda}\right). \end{aligned}$$

are then compatible; and the functions $w_k(k=1, 2, 3, 4)$, so defined, all satisfy the equation

$$(11.3)\qquad \frac{\partial^2 w}{\partial u\partial v}=\frac{1}{H}\frac{\partial}{\partial v}\log\frac{\alpha}{\lambda}\frac{\partial w}{\partial u}+H\frac{\partial}{\partial u}\log\frac{\beta}{\lambda}\frac{\partial w}{\partial v}\cdot$$

The five functions

$$(11.4)\qquad \begin{aligned} &\zeta_1^i=\frac{w_i}{(w\,|\,w)^{1/2}}, \qquad i=1,2,3,\\ &\omega_1=\frac{w_4}{(w\,|\,w)^{1/2}}, \qquad \lambda_1=\frac{1}{(w\,|\,w)^{1/2}} \qquad [(w\,|\,w)=w_1^2+w_2^2+w_3^2], \end{aligned}$$

are solutions of the equation

$$(11.5)\qquad \begin{aligned} \frac{\partial^2\lambda_1}{\partial u\partial v}=\frac{\partial}{\partial v}\log\frac{\alpha\bar{t}}{(w\,|\,w)^{1/2}}\frac{\partial\lambda_1}{\partial u}\\ +\frac{\partial}{\partial u}\log\frac{\beta\bar{s}}{(w\,|\,w)^{1/2}}\frac{\partial\lambda_1}{\partial v}-\frac{\left(w\times\dfrac{\partial w}{\partial u}\,\middle|\,w\times\dfrac{\partial w}{\partial v}\right)}{(w\,|\,w)^2}\lambda_1\,. \end{aligned}$$

From (11.4), $(\zeta_1|\zeta_1)\equiv 1$. Thus the functions ζ_1 and ω_1 can be considered as the tangential coördinates of a net N_1 whose tangential equation will be (11.5). Moreover, since (11.2) assumes the form (8.5) when the functions w are replaced by ζ_1, ω_1 and λ_1, as indicated by (11.4), the net N_1 is an F transform of N. The analytic work here is the same as in §3, but the inter-

* ζ^1, ζ^2, ζ^3 are the three ordered components of ζ (cf. (8.1)).

pretation now is that these quadratures determine ζ_1 and ω_1 only to within a perspective transformation (cf. equations (10.2)).

FUNDAMENTAL THEOREM III. *A solution* λ *of the tangential equation* (8.3) *of a net* N, *and a function* $H(u, v)$ *which satisfies* (11.1) *determine nets* N_1 *to within a perspective transformation which are* F *transforms of* N *by means of* λ; *the invariant* H *of these transformations* F *is the given function* $H(u, v)$.

12. **Triads.** We have seen that if N_1 and N_2 are F transforms of a net N (with (8.3) as its tangential equation) and if N_1 and N_2 are themselves in relation F, then the nets N, N_1, N_2 form either a harmonic or a conjugate triad.* From the nature of the function λ† of (8.5), which we say is the solution of the tangential equation of N used in the transformation F, we see that if

(i) the same λ is used in (8.5) to obtain N_1 and N_2 as F transforms of N, the three nets form a conjugate triad; and if

(ii) different solutions λ are used, the three nets form a harmonic triad.

We have thus the following *dual* relations:

If N_1, N_2, N_3 are three nets in a conjugate[harmonic] triad, the three transformations F have the same conjugate[harmonic] congruence but different harmonic[conjugate] congruences; any two of the nets are obtained from the third by means of the same solution $\lambda[\theta]$ of its tangential[point] equation, but by different solutions $\theta[\lambda]$ of its point[tangential] equation.

In the same way the invariants C and H, and radial and perspective transformations are duals of one another.

The methods used in proving the results embodied in the theorem of §6 are applicable to the invariant H also. Consequently we now have the complete

THEOREM IV. *If the product of two transformations* F *is a transformation* F, *the three nets in question form either a conjugate or a harmonic triad. In either case, the invariants* C *and* H *of the product transformation are equal respectively to the product of the invariants* C *and to the product of the invariants* H *of the given two transformations.*

As in §6, we also have, dually, the

COROLLARY. *If two transformations* F *of a net by the same solution of its tangential equation have equal invariants* H, *the two* F *transforms are per-*

* Cf. § 6, above.

† I.e. $\lambda=(\zeta\,|\,x')$, cf. § 8. Here λ is the fourth tangential coördinate of N', the net parallel to N, whose point coördinates are the direction parameters of the conjugate congruence of the transformation.

spective transforms of one another; and, conversely, if two non-perspective F transformations with different harmonic congruences of a net N are perspective transforms of one another, the two transformations F are by means of the same solution of the tangential equation of N and have equal invariants H.

13. Transformations F in homogeneous tangential coordinates. Analytically, the study of transformations F, whether in terms of homogeneous point coördinates, or in terms of homogeneous tangential coördinates, is the same.* The work and fundamental theorem of §7 need therefore only to be dually interpreted to obtain the facts for transformations F in terms of homogeneous tangential coördinates.

14. The invariants C and H as products of invariants. The transformation F as set up by Eisenhart† is the product of a parallel transformation P_1, a radial transformation R_2, and another parallel transformation P_3; i.e.,

$$N \xrightarrow{P_1} N' \xrightarrow{R_2} N_1' \xrightarrow{P_3} N_1.$$

Graustein‡ has shown that the invariant C of the product transformation F is equal to the product of the invariants (C) of these factor transformations.

Consider now the case of the invariant H. Since P_1 and P_3 are parallel, i.e., perspective, transformations, $H_1 = H_3 = 1$. For R_2, $C_2 = 1$, and hence, from (9.2), $H_2 = e_1' g'/(g_1' e')$, the quantities bearing on N_1' and N'. From (1.2)

$$e' = he, \quad g' = lg.$$

Since

$$\frac{\partial x_1'}{\partial u} = -\frac{h}{t}\frac{\partial x_1}{\partial u}, \qquad \frac{\partial x_1'}{\partial v} = -\frac{l}{s}\frac{\partial x_1}{\partial v}, \text{ §}$$

we also have

$$e_1' = -\frac{h}{t} e_1, \qquad g_1' = -\frac{l}{s} g_1.$$

Hence, since $C = t/s$,

$$H_2 = \frac{e_1 g}{g_1 e} \cdot \frac{1}{C} \cdot$$

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* Eisenhart, T. S., §§ 37, 38 and §§ 51, 52.

† Eisenhart, T. S., § 15.

‡ Cf. W. C. Graustein, *An invariant of a general transformation of surfaces*, § 5.

§ Eisenhart, T. S., § 15.

As a result

$$H = H_1 H_2 H_3.$$

THEOREM V. *The product of the invariants H of the parallel, radial, and a second parallel transformation into which a transformation F can be factored (in Eisenhart's way) is equal to the invariant H of the transformation F.*

The transformation F as considered by Jonas* was built up of a radial transformation R_1, a parallel transformation P_2, and another radial transformation R_3; i.e.,

$$N \xrightarrow{R_1} \overline{N} \xrightarrow{P_2} \overline{N}' \xrightarrow{R_3} N_1.$$

In this case also Graustein† has shown that the invariant C of the transformation F is equal to the product of those of R_1, P_2, R_3.

The invariant H for P_2, i.e. H_2, is unity since P_2 is a perspective transformation. For R_1 and R_3, $C_1 = C_3 = 1$. Hence, from (9.2),

$$H_1 = \frac{\bar{e}g}{\bar{g}e}, \quad H_3 = \frac{e_1 \bar{g}'}{g_1 \bar{e}'} \cdot$$

But $\bar{e}' = t\bar{e}$, $\bar{g}' = s\bar{g}$.‡ Thus, again,

$$H = H_1 H_2 H_3.$$

THEOREM VI. *The product of the invariants H of the radial, parallel and second radial transformation into which a transformation F can be factored (in Jonas' way) is equal to the invariant H of the transformation F.*

We are led to the conclusion from these facts that the methods of Eisenhart and Jonas are *duals* of one another.

III. NETS OF SPECIAL TYPE

15. **Nets with equal invariants.** The point equation of the net N_1 determined in §1 as an F transform of the net N having (1.1) as its point equation is

$$\frac{\partial^2 \theta_1}{\partial u \partial v} = \frac{\partial}{\partial v} \log \left(\frac{a t}{\theta'}\right) \frac{\partial \theta_1}{\partial u} + \frac{\partial}{\partial u} \log \left(\frac{bs}{\theta'}\right) \frac{\partial \theta_1}{\partial v} \cdot \text{§} \tag{15.1}$$

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* Cf. footnote on Jonas, Introduction.

† W. C. Graustein, *An invariant of a general transformation of surfaces.*

‡ Cf. Eisenhart, T. S., § 16, (21).

§ Eisenhart, T. S., § 15.

If the point equation of $N(1.1)$ has equal point invariants,*

$$\frac{\partial^2}{\partial u \partial v} \log\left(\frac{a}{b}\right) = 0,$$

and conversely. Thus N_1 will also have equal point invariants if and only if

$$\frac{\partial^2 \log C}{\partial u \partial v} = 0\ ;\ \text{i.e.}\ \ C = \frac{U(u)}{V(v)},$$

where U is a function of u alone and V of v alone.

The net N whose tangential equation is (8.3) has equal tangential invariants† if and only if

$$\frac{\partial^2}{\partial u \partial v} \log\left(\frac{\alpha}{\beta}\right) = 0.$$

Hence N_1 having (11.6) as its tangential equation will also have equal tangential invariants if and only if

$$\frac{\partial^2 \log H}{\partial u \partial v} = 0;\ \ \text{i.e.}\ \ H = \frac{U(u)}{V(v)}\cdot$$

THEOREM VII. *An F transform of a net N having equal point [tangential] invariants will also have equal point [tangential] invariants if and only if the invariant $C[H]$ of the transformation is of the form $U(u)/V(v)$.*

16. **Transformations F with constant invariants.** Consider two transformations F of a net N by means of the same solution θ of its point equation. Let the two invariants C of these transformations, C_1 and C_2, be constant. Equation (3.2) yields

$$\left(\frac{1}{C_i} - 1\right)\frac{\partial^2}{\partial u \partial v} \log\left(\frac{\theta}{a}\right) + (1 - C_i)\frac{\partial^2}{\partial u \partial v} \log\left(\frac{\theta}{b}\right) = 0 \qquad (i = 1,2). \tag{16.1}$$

Thus, either

(i) $$\frac{\partial^2}{\partial u \partial v} \log\left(\frac{\theta}{a}\right) = 0 \quad \text{and} \quad \frac{\partial^2}{\partial u \partial v} \log\left(\frac{\theta}{b}\right) = 0\ ;$$

or

(ii) $$\begin{vmatrix} \dfrac{1}{C_1} - 1 & 1 - C_1 \\ \dfrac{1}{C_2} - 1 & 1 - C_2 \end{vmatrix} = 0.$$

* Eisenhart, T. S., § 6.

† Eisenhart, T. S., § 53.

If (i) obtains, N and the two F transforms have equal point invariants. If, then, N has unequal point invariants, (ii) must hold; i.e.,

$$(1 - C_1)(1 - C_2)(C_1 - C_2) = 0.$$

From this result and its dual we are led to

THEOREM VIII. *If, in a harmonic[conjugate] triad of nets with unequal point[tangential] invariants, the invariants $C[H]$ of the three transformations F are constant, at least one of the transformations is radial[perspective].*

However, if (ii) does not hold, (i) must, and N will have equal point invariants. The F transforms will also have equal point invariants (cf. §15).

THEOREM IX. *If a net N admits of two non-radial [non-perspective] transformations F by means of a given solution $\theta[\lambda]$ of its point [tangential] equation with constant but unequal invariants $C[H]$, the net N has equal point [tangential] invariants; and the F transforms also have equal point [tangential] invariants.*

17. **Transformations K and Ω.** A transformation F for which $C = -1$ is called a *transformation* K,* and one for which $H = -1$, a *transformation* Ω.† For a transformation K, equation (3.2) yields

$$\frac{\partial^2}{\partial u \partial v} \log\left(\frac{a}{b}\right) = 0\,;$$

and for a transformation Ω, equation (11.1) becomes

$$\frac{\partial^2}{\partial u \partial v} \log\left(\frac{\alpha}{\beta}\right) = 0.$$

Thus we have the known fact that two nets in relation $K[\Omega]$ have equal point[tangential] invariants.

If N has equal point invariants its point equation is of the form

$$\frac{\partial^2 \theta}{\partial u \partial v} = \frac{\partial \log \psi}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log \psi}{\partial u}\frac{\partial \theta}{\partial v}. \tag{17.1}$$

Equation (3.2) for a constant invariant C of an F transformation of such a net yields

* Eisenhart, T. S., § 25.
† Eisenhart, T. S., § 53.

$$\left(\frac{1}{C} - C\right)\frac{\partial^2}{\partial u \partial v}\log\left(\frac{\theta}{\psi}\right) = 0. \tag{17.2}$$

We may say that in general

$$\frac{\partial^2}{\partial u \partial v}\log\left(\frac{\theta}{\psi}\right) \neq 0.$$

This fact and its dual leads to

THEOREM X. *A non-radial[non-perspective] transformation F of a net with equal point[tangential] invariants having a constant invariant $C[H]$ is, in general, a transformation $K[\Omega]$.*

18. **The product of the invariants C and H.** From (9.2) we see that if $CH=1$, we have

$$\frac{e_1}{g_1} = \frac{e}{g}\,. \tag{18.1}$$

Hence

THEOREM XI. *If two nets correspond by a transformation F for which the product of the invariants C and H is unity, the surfaces of the nets are so mapped that their asymptotic lines correspond. Conversely, if the surfaces of two nets in relation F are so mapped that their asymptotic lines correspond, the product of the invariants C and H of the transformation F is unity.*

From (9.2) we conclude

THEOREM XII. *If the parameters of one of two nets in relation F are isothermal-conjugate,* those of the other net are also isothermal-conjugate if and only if the product of the invariants C and H of the transformation F is unity.*

We can go a step farther. A *net* is isothermal-conjugate if, when parametric, $e/g = U(u)/V(v)$.† Consequently, using Theorem XII and (9.2) we have

THEOREM XIII. *If the nets N and N_1 in relation F have three of the following four properties, they have the fourth also:*

(a) *N and N_1 have equal point invariants;*
(b) *N and N_1 have equal tangential invariants;*
(c) *N is isothermal-conjugate;*
(d) *N_1 is isothermal-conjugate.*

* Eisenhart, *Differential Geometry*, pp. 198–199.
† Eisenhart, *Differential Geometry*, loc. cit.

If (a)[(b)] *holds and* (c) *and* (d) *both hold, and one of the nets N and N_1 has equal tangential*[*point*] *invariants, then the other net also has equal tangential*[*point*] *invariants.*

We may write (9.2) in the form

$$CH = \frac{e_1g_1}{D_1^2}\cdot\frac{D^2}{eg}\cdot\frac{g^2D_1^2}{g_1^2D^2} = \frac{K_1}{K}\cdot\left(\frac{gD_1}{g_1D}\right)^2, \tag{18.2}$$

where K and K_1 are the Gaussian curvatures of the surfaces carrying N and N_1.

THEOREM XIV. *If the nets N and N_1 in relation F are real and the parameters are real, their surfaces have their Gaussian curvatures of the same or opposite sign according as the product CH of the transformation is positive or negative.*

IV. TRANSFORMATIONS OF RIBAUCOUR

19. *O*-nets and conjugate normal congruences. The curves of a net N form an orthogonal system; i.e., N is an O-net, if and only if they are the lines of curvature of the surface of N. The point equation of the O-net of a surface is

$$\frac{\partial^2\theta}{\partial u\partial v} = \frac{\partial \log E^{1/2}}{\partial v}\frac{\partial\theta}{\partial u} + \frac{\partial \log G^{1/2}}{\partial u}\frac{\partial\theta}{\partial v}, \tag{19.1}$$

and its tangential equation is

$$\frac{\partial^2\lambda}{\partial u\partial v} = \frac{\partial}{\partial v}\log\frac{e}{E^{1/2}}\frac{\partial\lambda}{\partial u} + \frac{\partial}{\partial u}\log\frac{g}{G^{1/2}}\frac{\partial\lambda}{\partial v}. \tag{19.2}$$

The congruence of normals to a surface is conjugate to its O-net. In fact the spherical representation of this O-net N^* serves as a parallel net N' whose coördinates are direction parameters of this normal conjugate congruence. Let ζ be the direction cosines of the normals:

$$\frac{\partial\zeta}{\partial u} = -\frac{e}{E}\frac{\partial x}{\partial u}, \qquad \frac{\partial\zeta}{\partial v} = -\frac{g}{G}\frac{\partial x}{\partial v}. \tag{19.3}$$

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[*] The surface of N is assumed to be neither a sphere nor a developable.

If θ is an arbitrary solution of (19.1), equations

$$\frac{\partial p}{\partial u} = -\frac{e}{E}\frac{\partial \theta}{\partial u}; \qquad \frac{\partial p}{\partial v} = -\frac{g}{G}\frac{\partial \theta}{\partial v} \tag{19.4}$$

are compatible, and a function p so defined is a solution of the point equation of the spherical representation N' of the O-net N. Hence an arbitrary F transform N_0 of N along its normal conjugate congruence has the coördinates

$$x_0 = x - \frac{\theta}{p}\zeta. \tag{19.5}$$

For the surface of N_0,

$$F_0 = \frac{\partial \theta_0}{\partial u}\frac{\partial \theta_0}{\partial v}, \tag{19.6}$$

where

$$\theta_0 = -\frac{\theta}{p}. \tag{19.7}$$

If $\partial\theta_0/\partial u = \partial\theta_0/\partial v = 0$, θ_0 is constant, and the net N_0 of (19.5) is parallel to N.

If $\partial\theta_0/\partial u = 0$, $\partial\theta_0/\partial v \neq 0$, equations (19.7) and (19.4) show that $\partial\theta/\partial u = 0$, $\partial\theta/\partial v \neq 0$. Thus θ_0 and θ are functions of v alone. In this case we have for the directions of the curves of N_0*

$$\begin{aligned} \frac{\partial x_0}{\partial u} &= -\left(\frac{e\theta_0}{E} - 1\right)\frac{\partial x}{\partial u}, \\ \frac{\partial x_0}{\partial v} &= \left(\frac{g\theta_0 - G}{Gp}\right)\left(\zeta\frac{\partial \theta}{\partial v} - p\frac{\partial x}{\partial v}\right). \end{aligned} \tag{19.8}$$

Such a transformation is of a type studied by Graustein† and termed by him a *semi-parallel map*.

THEOREM XV. *A necessary and sufficient condition that an F transform of an O-net along its normal conjugate congruence be an O-net is that the transformation be either parallel or semi-parallel.*

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* Eisenhart, T. S., § 15, (19).

† W. C. Graustein, *Semi-parallel maps of surfaces*, Annals of Mathematics, (2), vol. 27 (1926), p. 271.

20. **Transformations R.** If $N':(x')$ is an arbitrary net parallel to the O-net N, the equations

$$(20.1)\qquad \frac{\partial\theta}{\partial u} = \left(x' \middle| \frac{\partial x}{\partial u}\right), \qquad \frac{\partial\theta}{\partial v} = \left(x' \middle| \frac{\partial x}{\partial v}\right)$$

are compatible, and θ will satisfy (19.1). The function

$$(20.2)\qquad \theta' = \frac{(x' \mid x')}{2}$$

is a "corresponding" solution of N' in the sense of §1. Thus we may obtain a net N_1: (x_1)

$$(20.3)\qquad x_1 = x - \frac{2\theta}{(x' \mid x')}x'$$

as an F transform of N. Here N_1 is also an O-net.

As a matter of fact, the surfaces of the O-nets N and N_1 are the sheets of the envelope of a two-parameter system of spheres, the curves of N and N_1 being the loci of the points of contact of the spheres.* This transformation F of N into N_1, as indicated by (20.3), is termed a *transformation of Ribaucour*, or briefly, a *transformation R*.

The invariants C and H of a transformation R enter in the symmetrical relations

$$(20.4)\qquad C^2 = \frac{E_1G}{G_1E}, \qquad CH = \frac{e_1g}{g_1e}, \qquad H^2 = \frac{\mathcal{E}_1\mathcal{G}}{\mathcal{G}_1\mathcal{E}}\cdot\dagger$$

The points with coördinates ζ and ζ_1, the direction cosines of the normals to N and N_1, trace O-nets on the unit sphere which are in relation F.‡ The point equations of these nets on the unit sphere are the same as their tangential equations and are also equal to the tangential equations of the nets N and N_1. Thus

THEOREM XVI. *The invariant H of a transformation R is equal to the invariant $C(=H)$ of the transformation F existing between the spherical representations of the nets in the relation R.*

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* Eisenhart, T. S., §§ 68–72.

† $\mathcal{E}, \mathcal{F}, \mathcal{G}$ are the coefficients of the square of the linear element of the spherical representation of the surface of S.

‡ Eisenhart, T. S., § 78, (11).

21. **Applications.** We return to the equations (20.4), and exclude radial and perspective transformations.

If the transformation R is also K, $C=-1$ and

$$\frac{E_1}{E}=\frac{G_1}{G} \tag{21.1}$$

and conversely. Since N and N_1 have equal point invariants (cf. §17), their surfaces are isothermic. Conversely, if the surfaces of O-nets in relation R are isothermic, we can choose parameters so that (21.1) holds. Moreover, (21.1) is the condition that the map be conformal.

The following theorems are thus readily obtained:

THEOREM XVII. *A necessary and sufficient condition that the surfaces of O-nets in relation R be conformally mapped is that both surfaces be isothermic. The transformation is then also K.**

THEOREM XVIII. *A necessary condition that the two surfaces whose O-nets are in relation R be isometrically mapped is that the transformation be also K. Both surfaces are then isothermic.*

THEOREM XIX. *A necessary and sufficient condition that the spherical representations of two surfaces whose O-nets are in relation R be conformally mapped is that the transformation be also Ω.*

THEOREM XX. *A necessary condition that the spherical representations of two surfaces whose O-nets are in relation R be isometrically mapped is that the transformation R be also Ω.*

Finally we have

THEOREM XXI. *If two surfaces whose O-nets are in relation R are mapped conformally (or isometrically) and if their spherical representations are also so mapped, the transformation R is both K and Ω; and the surfaces and their spherical representations are isothermic. Conversely, if these surfaces and their spherical representations are isothermic, the surfaces are conformally mapped and so also are their spherical representations, and the transformation R is both K and Ω. In these cases of a transformation R being both K and Ω, the asymptotic lines of both surfaces correspond.*†

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* Theorem of Cosserat, Eisenhart, T. S. § 82 and footnote (61).

† Cf. Theorem XI.

If the O-net of a minimal surface is parametric,

$$\frac{E}{G} = -\frac{e}{g}. \tag{21.2}$$

Thus we are led to

THEOREM XXII. *When the O-nets of two minimal surfaces are in relation R, the invariants C and H of the transformation are equal; and, conversely, if the invariants C and H of a transformation R are equal and the surface of one of the O-nets is minimal, so is the surface of the second.*

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2

THE FOUNDATIONS OF A THEORY IN THE CALCULUS OF VARIATIONS IN THE LARGE*

BY

MARSTON MORSE

Introduction

The conventional object in a paper on the calculus of variations is the investigation of the conditions under which a maximum or minimum of a given integral occurs. Writers have accordingly done little with extremal segments that have contained more than one point conjugate to a given point. An extended theory is needed for several reasons.

One reason is that in applying the calculus of variations to geodesics, or to that very general class of dynamical systems or differential equations which may be put in the form of the Euler equations, *it is by no means a minimum or a maximum that is always sought.* For example, if in the problem of two bodies we make use of the corresponding Jacobi principle of least action† the ellipses which thereby appear as extremals always have pairs of conjugate points on them, and do not accordingly give a minimum to the integral relative to neighboring closed curves, so that *no example of periodic motion would be found by a search for a minimum* of the Jacobi integral. In general if one is looking for extremals joining two points or periodic extremals deformable into a given closed curve, the a priori expectation, as justified by the results of this paper, in general problems, would seem to be that more solutions would not give an extremum than would give an extremum.

A second reason for the study of extremal segments and periodic extremals that do not furnish an extremum for the integral is that *if the ultimate object sought is an extremum, the existence of such extrema is tied up "in the large" with the existence of extremals which do not furnish extrema.* It is one of the purposes of this paper to show the relations in the large between all sorts of extremals joining two fixed points, or deformable into a given closed curve.

A first type of a priori existence theorem is Hilbert's theorem concerning

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† See P. Appell, *Traité de Mécanique Rationnelle*, Paris, 1919, vol. 1, pp. 547–548.

the absolute minimum.* Reference should be given to the more recent work of Tonelli,† also on the absolute minimum. Other references to studies of this sort will be found in the standard treatises. Birkhoff‡ effectively departed from the study of the absolute minimum alone, when he stated his "minimax principle," and applied it to find closed trajectories that do not give a minimum to his "least action" integral. The present writer§ has shown that Birkhoff's points of minimum and minimax appear as two types of critical points among $n+1$ such types, and has replaced Birkhoff's inequality relation by n inequality relations and one equality relation. One of the objects of the present paper is to show how these $n+1$ relations between critical points can be translated into relations between different types of extremals.

It was necessary to develop for the first time a complete parallelism between types of critical points and types of extremals. It was found that the type of an extremal segment whose ends were not conjugate was completely determined by the number of mutual conjugate points on the segment. In the case where the end points were conjugate it was necessary to bring in the envelope theory.

Turning to periodic extremals it was found, even in the most general cases, that conjugate points would not serve to determine the type of a periodic extremal g, but that other relations of g to neighboring extremals had to be brought in. A periodic extremal was called *degenerate* if the corresponding Jacobi differential equation possessed periodic solutions not identically zero. It is shown for the first time how a parameter may be introduced into the integrand in such a fashion that the degenerate periodic extremal disappears and non-degenerate periodic extremals, or no periodic extremals, take its place.

This theory is brought to a head in two applications, one "in the large," giving relations between different types of extremals, and one "in the small," showing how the type of a given extremal may be characterized in a third way, in terms of the possibility or impossibility of deformations of m-parameter families of neighboring extremals into families of lower dimensionality. This deformation theory was made possible by the application

* Bolza, *Vorlesungen über Variationsrechnung*, 1909, pp. 419–437. Further references to Bolza will be indicated by the letter B.

† Tonelli, *Fondamenti di Calcolo delle Variazioni*, vol. 2.

‡ Birkhoff, *Dynamical systems with two degrees of freedom*, these Transactions, vol. 18 (1917), p. 240.

§ Marston Morse, *Relations between the critical points of a real function of n independent variables*, these Transactions, vol. 27 (1925), pp. 345–396.

of certain powerful theorems of analysis situs. It is believed that this deformation theory will serve as the basis for an even more extended theory "in the large."

PART I. THE TYPE NUMBER OF AN EXTREMAL SEGMENT

1. **The integrand $F(x, y, \dot{x}, \dot{y})$ and the function $J(v_1, \cdots, v_n)$.** Let (x, y) be any point in an open two-dimensional region S of the x, y plane. Let there be given a function $F(x, y, \dot{x}, \dot{y})$ of class C''' (B, loc. cit., p. 193), and positively homogeneous in $\dot{x}$ and $\dot{y}$ of dimension one, for (x, y) in S and $\dot{x}$ and $\dot{y}$ any two numbers not both zero. Corresponding to the calculus of variations problem in the parametric form with integral J, and with $F(x, y, \dot{x}, \dot{y})$ as the integrand, let there be given an extremal g of class C''' (B, p. 191), without multiple points, passing from a point A to a point B, and such that along g we have (B, p. 196)

$$F_1(x, y, \dot{x}, \dot{y}) > 0. \tag{1}$$

Let the arc length along g, measured from A toward B, be denoted by u. Let

$$u_1, u_2, \cdots, u_n \tag{2}$$

be n values of u increasing with their subscripts, and corresponding to points on g between A and B. Denote the value of u at A by u_0, and its value at B by u_{n+1}. We suppose the points (2) so chosen on g that no one of the closed segments of g bounded by successive points of the set

$$u_0, u_1, \cdots, u_{n+1} \tag{3}$$

contains a conjugate point of either end point of that segment. Let there be given n short arcs of class C''', say $h_1, h_2, \cdots, h_n$, passing respectively through the points of g at which u takes on the values (2), arcs not tangent to g at these points. Let positive senses be assigned to $h_1, h_2, \cdots, h_n$ in such a fashion that the positive tangent to g at $u = u_i$ has to be turned through a positive angle less than π to coincide with the positive tangent to h_i at the same point. Let v_i be the arc length measured along h_i in h_i's positive sense from the point $u = u_i$ on g. We regard (u_i, v_i) as representing the point on h_i at the distance v_i from g.

If each v_i be sufficiently small in absolute value the points

$$A, (u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n), B \tag{4}$$

can be successively joined by unique extremal segments neighboring g.

Let the integral J taken along this broken extremal joining A to B be denoted by

(5) $$J(v_1, \cdots, v_n).$$

Corresponding to the extremal g the function (5) will be shown to have a *critical point* for $(v_1, \cdots, v_n) = (0, 0, \cdots, 0)$, that is *a point at which all of its partial derivatives are zero.* We will also investigate the terms of second order in the expansion of the function (5) about $(0, 0, \cdots, 0)$.

2. **A transformation of the problem.** We introduce the following lemma which simplifies the problem.

LEMMA. *It is possible to make a transformation T from the (x, y) plane to the (u, v) plane with the following properties:*

(A) *Under T the extremal g is carried into a portion γ of the u axis bounded by $u = u_0$ and $u = u_{n+1}$.*

(B) *The transformation establishes a one-to-one correspondence between a suitably chosen region of the (x, y) plane enclosing g, and a region R_1 of the (u, v) plane enclosing γ.*

(C) *The transformation carries the arcs $h_1, h_2, \cdots, h_n$ into straight line segments on which $u = u_1, u_2, \cdots, u_n$, respectively.*

(D) *The transformation is representable in the form*

(1) $$x = x(u, v), \quad y = y(u, v)$$

where $x(u, v), y(u, v)$ are of class C''' in R_1 and possess there a positive jacobian.

(E) *The transformation preserves arc lengths along g and $h_1, h_2, \cdots, h_n$.*

In the first preparation of this paper the complete details of the proof of this lemma were given, but second thought makes it seem not too much to leave to the reader.

3. **The integral in the (u, v) plane.** Under the transformation of the preceding lemma the integrand $F(x, y, \dot{x}, \dot{y})$ can be replaced by a new integrand $G(u, v, \dot{u}, \dot{v})$, where $\dot{x}, \dot{y}, \dot{u}, \dot{v}$ stand for derivatives of x, y, u, v, respectively, with respect to a parameter t (B, p. 344). We are concerned with properties of the extremal segment γ, namely,

(1) $$v \equiv 0, \quad u_0 \leqq u \leqq u_{n+1},$$

which corresponds under the preceding transformation T to the extremal g in the (x, y) plane. For the present we need only consider curves on which $\dot{u} > 0$, and set

(2) $$G(u, v, 1, v') = f(u, v, v'), \quad v' = \frac{dv}{du},$$

thus defining $f(u, v, v')$ for all points (u, v) neighboring γ, and all numbers v'. For this domain $f(u, v, v')$ is of class C'''. The points which are enumerated in §1, (4), here correspond to points whose actual coördinates are

$$(3)\qquad (u_0, 0),\ (u_1, v_1),\ \cdots,\ (u_n, v_n),\ (u_{n+1}, 0).$$

Subject to the limitation that we deal here only with curves on which $\dot{u}>0$, the integral J of §1 becomes here

$$(4)\qquad J = \int_{u_0}^{u_{n+1}} f(u, v, v')du,$$

and the function

$$(5)\qquad J(v_1, \cdots, v_n)$$

is the value of the integral (4) taken along the successive extremals joining the successive points of (3), varying the coördinates v_i, but holding all coördinates u_i fast. It should be expressly noted that the function (5) *is not a new function set up for the first time in the* (u, v) *plane, but that it is identical with the function* $J(v_1, \cdots, v_n)$ *defined in* §1.

In terms of $f(u, v, v')$, and for the extremal γ given by (1), we define the functions $P(u)$, $Q(u)$, and $R(u)$ in the usual way (B, p. 55). These functions are of class C' along γ. Because of the assumption that $F_1>0$ along g, it follows that

$$(6)\qquad R(u) > 0$$

along γ. The Jacobi differential equation corresponding to the extremal segment γ will be written in the well known form

$$(7)\qquad Rw'' + R'w' + (Q' - P)w = 0,$$

with w the dependent variable, and u the independent variable. *We are always going to write J. D. E. for* (7).

4. **The second partial derivatives of $J(v_1, \cdots, v_n)$.** We shall see presently that $J(v_1, \cdots, v_n)$ has a critical point when $(v_1, \cdots, v_n) = (0, \cdots, 0)$. To determine the nature of this critical point we proceed to the determination of the second partial derivatives of $J(v_1, \cdots, v_n)$. To that end we represent the family of extremals which join the points

$$(1)\qquad (u_i, v_i),\ (u_{i+1}, v_{i+1}) \qquad (i = 0, 1, \cdots, n)$$

in the form

$$(2)\qquad v = r^i(u, v_i, v_{i+1}),\quad u_i \leqq u \leqq u_{i+1},$$

where it is understood that the coördinates u_i and u_{i+1} are held fast. The functions

$$r^i,\ r_u^i,\ r_{uu}^i \tag{3}$$

will be of class C' in all of their arguments for u on the interval in (2), and for v_i and v_{i+1} neighboring zero (B, p. 73 and p. 307).

We shall understand by $w_{\mu\nu}(u)$, a solution of the J. D. E., such that

$$w_{\mu\nu}(u_\mu) = 0,\quad w_{\mu\nu}(u_\nu) = 1 \quad (|\mu - \nu| = 1,\ \mu, \nu = 0, 1, 2, \cdots, n+1). \tag{4}$$

Because of the fact that the functions (2) satisfy the identities

$$\begin{aligned} v_i &\equiv r^i(u_i, v_i, v_{i+1}), \\ v_{i+1} &\equiv r^i(u_{i+1}, v_i, v_{i+1}), \end{aligned} \tag{5}$$

it follows that

$$\begin{aligned} w_{i+1,i}(u) &\equiv r^i_{v_i}(u, 0, 0), \\ w_{i,i+1}(u) &\equiv r^i_{v_{i+1}}(u, 0, 0). \end{aligned} \tag{6}$$

Differentiation of the integral J, and integration by parts in the usual way will now give

$$J_{v_i}(v_1, \cdots, v_n) = f_{v'}[u_i, v_i, r_u^{i-1}(u_i, v_{i-1}, v_i)] - f_{v'}[u_i, v_i, r_u^i(u_i, v_i, v_{i+1})]. \tag{7}$$

Here it is understood that

$$i = 1, 2, \cdots, n,\quad v_0 = 0,\quad v_{n+1} = 0. \tag{8}$$

From (7) we see that all the partial derivatives of $J(v_1, \cdots, v_n)$ are zero at $(0, \cdots, 0)$, and note that $J(v_1, \cdots, v_n)$ is of class C' in the neighborhood of $(0, \cdots, 0)$.

From (7) we see that

$$J_{v_i, v_j} = 0,\quad |i - j| \neq 1, 0. \tag{9}$$

The remaining second partial derivatives will be evaluated at $(0, \cdots, 0)$. Evaluation at the latter point will be indicated by a subscript zero preceding the partial derivative. We obtain the following results:*

$$\begin{aligned} {}_0J_{v_i, v_i} &= R(u_i)[r^{i-1}_{u, v_i}(u_i, 0, 0) - r^i_{u, v_i}(u_i, 0, 0)] \\ &= R(u_i)[w'_{i-1,i}(u_i) - w'_{i+1,i}(u_i)] \qquad (i = 1, 2, \cdots, n); \end{aligned} \tag{10}$$

* A. Dresden has given complete formulas for the second partial derivatives of the extremal integral (Bolza, p. 310). Use has not been made of this work, however, because the formulas given in the present paper need not have the general form given by Dresden, and do need a different notation.

$$
{}_0J_{v_i,v_{i+1}} = -R(u_i)w'_{i,i+1}(u_i) \qquad (i = 1, 2, \cdots, n-1); \tag{11}
$$

$$
{}_0J_{v_{i+1}\,v_i} = R(u_{i+1})w'_{i+1,i}(u_{i+1}) \qquad (i = 1, 2, \cdots, n-1). \tag{12}
$$

The last two partial derivatives are necessarily equal, as can be proved directly from the properties of the J. D. E.

The matrix whose elements are

$$
a_{i,j} = {}_0J_{v_i,v_j} \qquad (i, j = 1, 2, \cdots, n) \tag{13}
$$

has now been determined. Note first that all elements in the ith row of this matrix have the factor $R(u_i)$. Let this factor be removed from the ith row ($i=1, 2, \cdots, n$). We will write down the resulting matrix for a typical case, $n=5$.

$$
\left\| \begin{array}{lllll}
w'_{01}(u_1)-w'_{21}(u_1), & -w'_{12}(u_1) & , \quad 0 & , \quad 0 & , \quad 0 \\
w'_{21}(u_2) & , \; w'_{12}(u_2)-w'_{32}(u_2), & -w'_{23}(u_2) & , \quad 0 & , \quad 0 \\
\quad 0 & , \; w'_{32}(u_3) & , \; w'_{23}(u_3)-w'_{43}(u_3), & -w'_{34}(u_3) & , \quad 0 \\
\quad 0 & , \quad 0 & , \; w'_{43}(u_4) & , \; w'_{34}(u_4)-w'_{54}(u_4), & -w'_{45}(u_4) \\
\quad 0 & , \quad 0 & , \quad 0 & , \; w'_{54}(u_5) & , \; w'_{45}(u_5)-w'_{65}(u_5)
\end{array} \right\|
$$

5. **The rank of the extremal segment g.** We prove the following theorem:

THEOREM 1. *Let the function $J(v_1, \cdots, v_n)$ be set up for the extremal segment g, as described in* §1. *The matrix* a, *whose elements are*

$$
a_{ij} = {}_0J_{v_iv_j},
$$

is of rank n if the final point of g is not among the conjugate points of the initial point of g. Otherwise a *is of rank $n-1$.*

To prove this theorem let us turn to the (u, w) plane of the J. D. E., and in that plane join the successive points

$$
(u_0, 0)(u_1, c_1), \cdots, (u_n, c_n)(u_{n+1}, 0),
$$

by curve segments representing solutions of the J. D. E. The successive segments are of the form

$$
w = c_iw_{i+1,i}(u) + c_{i+1}w_{i,i+1}(u), \qquad u_i \leqq u \leqq u_{i+1}, \tag{1}
$$

where

$$
i = 0, 1, \cdots, n, \quad c_0 = 0, \quad c_{n+1} = 0. \tag{2}
$$

Let the curve obtained by combining the successive segments (1) be denoted by λ. A necessary and sufficient condition that the point $(u_0, 0)$ be conjugate to $(u_{n+1}, 0)$ is that among the curves λ there exist at least one, not

$w \equiv 0$, that has no corners at the junctions of its successive segments. The n conditions that λ have no corners at the junctions of these segments are

$$(3) \quad [c_{i-1}w'_{i,i-1}(u_i) + c_i w'_{i-1,i}(u_i)] - [c_i w'_{i+1,i}(u_i) + c_{i+1}w'_{i,i+1}(u_i) = 0],$$

where

$$(4) \qquad i = 1, 2, \cdots, n, \qquad c_0 = c_{n+1} = 0.$$

Equations (3) may be written in the form

$$(5) \qquad c_{i-1}w'_{i,i-1}(u_i) + c_i[w'_{i-1,i}(u_i) - w'_{i+1,i}(u_i)] - c_{i+1}w'_{i,i+1}(u_i) = 0,$$

subject again to the conditions (4).

Let the matrix of the coefficients of $(c_1, \cdots, c_n)$ in (5) be denoted by w and the value of the corresponding determinant by $|w|$. With the aid of (9), (10), (11), and (12) of §4 we obtain the following equation, giving the determinant $|a|$ of the theorem to be proved:

$$(6) \qquad |a| = R(u_1),\ R(u_2), \cdots, R(u_n) \,|\, w \,| \,.$$

For $|a|$, and accordingly $|w|$, to be zero, it is necessary and sufficient that the equations (5), subject to the conditions (4), admit a solution $(c_1, \cdots, c_n)$ in which the c_i's are not all zero. That is, for $|w|$ and $|a|$ to be zero, it is necessary and sufficient that $(u_0, 0)$ be conjugate to $(u_{n+1}, 0)$.

It remains to prove that if a is of rank less than n, its rank is exactly $n-1$. Suppose $|a|=0$. Then $(u_0, 0)$ is conjugate to $(u_{n+1}, 0)$. The point $(u_0, 0)$ cannot also be conjugate to $(u_n, 0)$, for otherwise $(u_n, 0)$ would be conjugate to $(u_{n+1}, 0)$, contrary to the original choice of $(u_1, \cdots, u_n)$. Now a consideration of the form of the minor A_{n-1} of the element a_{nn} of a shows that the vanishing of A_{n-1} is the condition that $(u_0, 0)$ be conjugate to $(u_n, 0)$. Hence $A_{n-1} \neq 0$, and a is of rank $n-1$. Thus the theorem is completely proved.

We denote by A_i $(i=1, 2, \cdots, n)$ the determinant obtained from a by striking out the last $n-i$ rows and columns of a, and set $A_0 = 1$.

COROLLARY 1. *A necessary and sufficient condition that* $A_i = 0$ $(i=1, 2, \cdots, n)$ *is that* $(u_0, 0)$ *be conjugate to* $(u_{i+1}, 0)$.

This follows at once from the form of A_i and from Theorem I.

COROLLARY 2. *The matrix* a *is in normal form.**

For if A_r and A_{r+1} $(0<r<n)$ were both zero, $(u_0, 0)$ would be conjugate

* Bôcher, *Introduction to Higher Algebra*, p. 59.

to both $(u_r, 0)$ and $(u_{r+1}, 0)$, contrary to the original restrictions on $(u_1, u_2, \cdots, u_n)$.

6. The type number of the extremal segment g. We prove the following theorem:

THEOREM 2.* *If the final end point B of the extremal segment g of §1 is not conjugate to its initial point A, but there are k points $(0 \leq k \leq n)$ on g between A and B conjugate to A, then the symmetric quadratic form*

$$\sum_{ij} {}_0J_{v_iv_j}v_iv_j \qquad (i, j = 1, 2, \cdots, n), \tag{1}$$

when reduced by a real non-singular linear transformation to squared terms only, will have k coefficients that are negative, and $n-k$ that are positive.

The number k is called the type number of the critical point $(0, \cdots, 0)$ *of $J(v_1, \cdots, v_n)$, and also the type number of g.*

To prove this theorem we shall make use of the well known fact (Bôcher, loc. cit., p. 147) that in a regularly arranged quadratic form the type number k equals the number of changes of sign in the sequence $A_0, A_1, \cdots, A_n$ (§5) where an A_i which is zero is counted as positive or negative at pleasure. To continue we adopt the method of mathematical induction. In the case $n=1$ we are concerned simply with the fixed end points $(u_0, 0)$ and $(u_2, 0)$, and an intermediate point $(u_1, 0)$. Here

$$A_1 = [w'_{0,1}(u_1) - w'_{2,1}(u_1)]R(u_1). \tag{2}$$

It is readily seen that this difference is negative or positive according to whether or not $(u_0, 0)$ possesses a conjugate point prior to $(u_2, 0)$. We now distinguish between the cases in which $A_{n-1}=0$, and $A_{n-1}\neq 0$.

Case I. $A_{n-1}\neq 0$. If we assume the validity of the theorem for the case where there are $n-1$ coördinates $v_1, \cdots, v_{n-1}$, we can conclude that there are as many changes in sign in the sequence $A_0, \cdots, A_{n-1}$ as there are conjugate points prior to $(u_n, 0)$. To determine whether there is a change of sign in passing from A_{n-1} to A_n, it is useful to consider again equations (5) of §5, unaltered except that the zero in the right hand member of the last equation is here replaced by

$$A_{n-1}/A_n \tag{3}$$

and the left hand member of the ith one of these equations is here to be multiplied by $R(u_i)$ $(i=1, 2, \cdots, n)$.

* The corresponding theorem in m dimensions has recently been discovered by the author.

The resulting equations can now be solved for the constants $c_1, \cdots, c_n$. In particular we obtain, by Cramer's rule,

$$c_n = \frac{A_{n-1}^2}{A_n^2} > 0.$$

The curve λ of §5 corresponding to these constants $(c_1, \cdots, c_n)$ has just one corner, and that at (u_n, c_n). The last equation of (5), §5, altered as we have said, now tells us that at this corner the slope of the first segment of λ exceeds or is less than the slope of the second segment of λ, according as the right hand member (3) is positive or negative. This result can be interpreted in terms of the J. D. E. to mean that there is, or is not, a conjugate point of $(u_0, 0)$ between $(u_n, 0)$ and $(u_{n+1}, 0)$, according as the quotient (3) is negative or positive. Thus, in case $A_{n-1} \neq 0$, the total number of changes of sign in the sequence $A_0, A_1, \cdots, A_n$ equals the total number of conjugate points of $(u_0, 0)$ prior to $(u_{n+1}, 0)$.

Case II. $A_{n-1} = 0$. In this case we make use of the fact that, if $A_{n-1} = 0$ in a regularly arranged non-singular quadratic form, then $A_{n-2} \neq 0$. Assuming then the validity of the theorem for the case where there are $n-2$ points between the end points of a given extremal we can conclude that there are as many changes of sign, say h, in the sequence $A_0, A_1, \cdots, A_{n-2}$ as there are conjugate points of $(u_0, 0)$ prior to $(u_{n-1}, 0)$. But since $A_{n-1} = 0$ it follows from Corollary 1, §5, that $(u_n, 0)$ is conjugate to $(u_0, 0)$ so that there are altogether $h+1$ conjugate points of $(u_0, 0)$ prior to $(u_{n+1}, 0)$. On the other hand it follows from the theory of regularly arranged quadratic forms that if $A_{n-1} = 0$, then A_{n-2} and A_n have opposite signs. Thus in the sequence A_0, $A_1, \cdots, A_n$, there are $h+1$ changes of sign, and the theorem is proved in Case II as well as in Case I.

Part II. Relations in the large between extremals joining A to B

7. **The integrand and region S.** In the developement of this chapter we do not wish to exclude the case where the end points of the extremal segment are conjugate. The complete treatment of an extremal segment whose ends are conjugate, both in our theory and in the classical theory, depends upon the nature of the singularity at B of the envelope of the extremals passing through A. Such a treatment, at least as developed so far, requires the assumption that the functions used be analytic.

I. *We therefore assume that the function $F(x, y, \dot{x}, \dot{y})$ is positively homogeneous of the first degree in $\dot{x}$ and $\dot{y}$, and analytic in all of its arguments, for x, y any point interior to an open region S of the (x, y) plane, and $\dot{x}$ and $\dot{y}$ any*

two numbers not both zero. We shall also assume that for the same arguments

$$F_1(x, y, \dot{x}, \dot{y}) > 0.$$

In S the differential equations of the extremals can be put in the Bliss* form,

$$\frac{dx}{ds} = \cos\theta, \quad \frac{dy}{ds} = \sin\theta, \quad \frac{d\theta}{ds} = H(x, y, \cos\theta, \sin\theta), \tag{1}$$

where

$$H(x, y, \dot{x}, \dot{y}) = \frac{F_{y\dot{x}} - F_{x\dot{y}}}{F_1[\dot{x}^2 + \dot{y}^2]^{3/2}}. \tag{2}$$

A necessary and sufficient condition that a solution g of (1) on which $(x, y, \theta) = (x_0, y_0, \theta_0)$ for some value of s, be identical as a set of points (x, y), with the solution on which $(x, y, \theta) = (x_0, y_0, \theta_0 + \pi)$ for some value of s, is that, on g,

$$H(x, y, \cos\theta, \sin\theta) + H(x, y, -\cos\theta, -\sin\theta) = 0. \tag{3}$$

II.† *We assume that* (3) *holds identically for every point* (x, y) *in* S, *and for every* θ. *This type of problem is termed reversible.*

8. **The regions** S_1, **and** R. We can choose between a great variety of assumptions that will serve as boundary conditions. The following perhaps are as simple as any.

III. *Let there be given a closed region* S_1, *consisting of points interior to* S, *bounded by a simple closed curve* β *consisting of a finite number of analytic arcs. We assume that this boundary* β *is extremal-convex*‡, *in the sense that the interior angles at the vertices of* β *shall be between* 0 *and* π, *and that an extremal tangent to an analytic arc* β' *of* β *at a point* P, *shall in the neighborhood of* P, *except for* P, *lie wholly on that side of* β' *which is exterior to* S_1.

IV.† *We assume further that the region* S_1 *is covered in a one-to-one manner by a proper field of extremals of the form*

$$x = h(u, v), \quad y = k(u, v), \tag{1}$$

where u *is the parameter, and* v *is the arc length measured along the extremals, where at every point* (u, v) *that corresponds to a point* (x, y) *in* S_1 *the functions* (1) *are single-valued and analytic in* u *and* v, *and*

* Bliss, these Transactions, vol. 7 (1906), p. 180.

† The author has recently removed hypotheses II and IV, assuming then that F is positive definite. Powerful results obtain, but the proofs are necessarily more difficult.

‡ Compare Bolza, loc. cit., pp. 276–278, and also Birkhoff, loc. cit., pp. 216–219.

$$D = \begin{vmatrix} h_u & h_v \\ k_u & k_v \end{vmatrix} \neq 0.$$

The region S_1 in the (x, y) plane will correspond in the (u, v) plane to a closed region R, bounded by a simple closed curve γ consisting of a finite number of analytic arcs, and again extremal-convex. When u is the independent variable, and v the dependent variable, the integral J can be put in the non-parametric form as in §3 with $f(u, v, v')$ the integrand. Because of the assumption of reversibility the non-parametric problem, with integrand $f(u, v, v')$, will here include all the extremals of the parametric form except those of the family $u=$constant. From our assumption regarding F_1, we have here that

$$f_{v'v'}(u, v, v') = 0,$$

for all points (u, v) in R, and any number v'. Concerning the region R, we can now establish the following lemma:

LEMMA. *Let a and b be, respectively, the minimum and maximum values of u on γ. Then γ consists of two arcs of the form*

$$v = A(u), \quad v = B(u), \qquad a \leqq u \leqq b,$$

where $A(u)$ and $B(u)$ are of class C, and analytic in u on the interval $a \leqq u \leqq b$, except for a finite set of values of u, while

$$A(u) < B(u), \qquad a < u < b, \tag{2}$$

$$A(a) = B(a), \quad A(b) = B(b). \tag{3}$$

Finally the interior points (u, v) of R are the points satisfying

$$A(u) < v < B(u), \qquad a < u < b. \tag{4}$$

To prove these statements let (u_0, v_0) be any interior point of R. There exist two numbers, v_1 and v_2, of such sort that the points (u, v) for which

$$u = u_0, \qquad v_1 < v < v_2, \tag{5}$$

include the point (u_0, v_0), but no points other than interior points of R, while the points (u_0, v_1) and (u_0, v_2) are on the boundary of R. Now the extremal-segment (5) is not tangent to γ at either of its ends, since γ is extremal convex. If γ has a vertex at (u_0, v_2), the two analytic arcs adjoining (u_0, v_2) must lie on opposite sides of $u=u_0$, at least in the neighborhood of (u_0, v_2), for otherwise the points of (5) in the neighborhood of (u_0, v_2) would not lie in R. A similar statement applies to (u_0, v_1). Thus, in

any case, there must exist functions $h(u)$ and $k(u)$ of class C, and constants a_1 and b_1, differing from u_0 by so little, that the points (u, v) satisfying

$$h(u) < v < k(u), \qquad a_1 < u < b_1, \tag{6}$$

are all interior points of R, while the curves

$$\begin{aligned} v &= h(u), & v_1 &= h(u_0), & a_1 &\leqq u \leqq b_1, \\ v &= k(u), & v_2 &= k(u_0), & a_1 &< u_0 < b_1, \end{aligned} \tag{7}$$

lie on the boundary of R.

We wish to show that these functions $h(u)$ and $k(u)$ can be extended in definition, as functions of class C, so that the preceding statements regarding (6) and (7) still hold true when $a_1 = a$ and $b_1 = b$. Whether this is true or not there certainly exist constants a_2 and b_2, such that

$$a \leqq a_2 < u_0 < b_2 \leqq b,$$

and functions $h(u)$ and $k(u)$ of class C, such that the curves

$$\begin{aligned} v &= h(u), & v_1 &= h(u_0), & & \\ v &= k(u), & v_2 &= k(u_0), & a_2 &\leqq u \leqq b_2, \end{aligned} \tag{8}$$

lie on the boundary of R, and the points (u, v) which satisfy

$$h(u) < v < k(u), \qquad a_2 < u < b_2, \tag{9}$$

are all interior points of R, while further, a_2 and b_2 are constants such that the interval in (8), and the corresponding interval in (9), cannot be expanded at either or both ends, and the preceding statements about (8) and (9) still hold true for these expanded intervals.

The following division into cases is exhaustive:

Case I $\quad a_2 = a, \quad b_2 = b$;
Case II $\quad a_2 = a, \quad b_2 < b$;
Case III $\quad a < a_2, \quad b_2 = b$;
Case IV $\quad a < a_2, \quad b_2 < b$.

Case I. In this case the theorem follows readily.

Case II. In this case we will prove that the inequality $b_2 < b$ is impossible. In the first place if we had $h(b_2) = k(b_2)$, the curve (8) would completely bound the region (9), a sub-region of R. This is impossible, since R consists of a single connected region. Hence $h(b_2) < k(b_2)$. The points (u, v) which satisfy

$$h(b_2) < v < k(b_2), \qquad u = b_2, \tag{10}$$

are limit points of points of (9), and are accordingly points of R. We distinguish again between two cases, Cases IIa and IIb.

In Case IIa we suppose the points of (10) are all interior points of R. In this case (8) and (9) will hold as stated for a slightly larger b_2, and thus we have a contradiction.

In Case IIb we suppose the points of (10) include at least one boundary point P of R. Such a point P cannot be a vertex of the boundary without the interior angle at the vertex being greater than π, contrary to an hypothesis. Neither can P be an ordinary point of the boundary, for in that case the boundary would have to be tangent to the extremal segment (10), a result which is again impossible, since the boundary is extremal-convex. Thus Case IIb leads to a contradiction under all circumstances, and is impossible.

Similarly, it can be proved that Cases III and IV are impossible. Case I alone is possible, and the theorem in italics follows readily.

9. **Further properties of the region R.** In R no extremals other than the extremals $u=$constant can be tangent to the extremals $u=$constant. Hence every extremal segment other than the extremal segments $u=$constant is representable in the form $v=M(u)$, where $M(u)$ is an analytic function of u, for u on the interval (§8, Lemma)

$$a \leqq u \leqq b,$$

or some sub-interval of that interval.

Let A and B be two distinct points of R, either interior to R, or on the boundary of R. *Any extremal joining A to B in R will have no points on the boundary of R, except possibly its end points A or B.* This follows readily from the extremal convex nature of the boundary (§8).

We will now prove *that two points A and B in R can be joined in R by at most a finite number of extremal segments.* If A and B are both on the same extremal segment $u=u_0$, that extremal segment is the only extremal segment in R which can join A to B. Suppose then that (u_0, v_0) and (u_1, v_1) are two points of R for which $u_0<u_1$, and which can be joined by an infinite set of extremal segments in R. Denote by m the angles in the (u, v) plane which the tangents to these extremals at (u_0, v_0) make with a parallel to the positive u axis. Take these angles between $-\pi/2$ and $\pi/2$.

Now there will be at least one limit angle m_0 of these angles m. The v coördinate of an extremal E_0 issuing from (u_0, v_0) with the angle m_0 can be continued as an analytic function of u until E_0 passes out of R or through (u_1, v_1). The extremal E_0 cannot pass out of R before passing through (u_1, v_1), because if E_0 did so pass out of R, extremals with angles m sufficiently

near m_0 would also pass out of R before passing through (u_1, v_1), which is impossible.

It would follow then from the envelope theory of extremals that all extremals, without exception, that issue from (u_0, v_0) with angles *neighboring* m_0 would pass through (u_1, v_1). Moreover, we could then prove that *all* extremals issuing from (u_0, v_0) with angles m for which

$$m_0 \leqq m < \pi/2, \tag{1}$$

would remain in R, at least until they passed through (u_1, v_1). For otherwise there would be a least upper bound $m_1 < \pi/2$, of angles m at which extremals in R issue from (u_0, v_0) and pass through (u_1, v_1). But the extremal issuing from (u_0, v_0) with the angle m_1 would lie entirely within R, except possibly for its end points, so that extremals issuing from (u_0, v_0) with angles slightly larger than m_1 would also lie in R, and pass through (u_1, v_1). Therefore no such least upper bound, m_1, for which $m < \pi/2$ exists. Thus all extremals which issue from (u_0, v_0) with angles m satisfying (1) must pass through (u_1, v_1) before passing out of R.

But an extremal issuing from (u_0, v_0) with an angle sufficiently near $\pi/2$ will pass out of R without passing through (u_1, v_1). From this contradiction we infer the truth of the statement to be proved.

10. **The function** $J(v_1, \cdots, v_n)$. *None of the extremal segments $u = u_0$ in R have a point on them conjugate to either end point.* To prove this let the J. D. E. in the Weierstrass form with independent variable $t = v$, be set up corresponding to the extremal $u = u_0$ in the (x, y) plane. This differential equation has as a solution the determinant D of §8, provided we set $u = u_0$ and let v vary. The absence of any conjugate points on $u = u_0$ follows from the non-vanishing of this determinant. From the absence of conjugate points in R on the extremals $u = u_0$, the "regularity" $(F_1 > 0)$, and analyticity of the problem, and the extremal convex nature of the boundary, it follows that *a positive constant e can be determined small enough to have the following properties. Any point of R for which $u = u_0$, can be joined to any point of R for which u differs from u_0 in absolute value by less than e, by a unique analytic extremal h lying entirely within R, excepting possibly its end points, and such that on h there are no pairs of conjugate points* (B, p. 307). The questions of uniformity arising can be met by the reader without too much difficulty.

Now let there be given two points A and B of R, not on the same extremal segment $u = u_0$. We are concerned with the extremals joining A to B, if any such exist. Let $u_0, u_1, \cdots, u_{n+1}$ be a set of increasing values of u of which successive values differ at most by the constant e of the preceding paragraph, and which are such that A lies on $u = u_0$, and B on $u = u_{n+1}$. Consistent with

this, let the coördinates of A and B be respectively (u_0, v_0) and (u_{n+1}, v_{n+1}), and $(v_1, \cdots, v_n)$ be variables such that the points

$$(u_1, v_1), \cdots, (u_n, v_n) \tag{1}$$

are all in R. Let the successive points of the set

$$(u_0, v_0),\ (u_1, v_1), \cdots, (u_{n+1}, v_{n+1}) \tag{2}$$

be joined by the unique extremal segments of the preceding paragraph, and let the integral J be evaluated along the resulting curve. If we hold the end points A and B fast, as well as the u coördinates of the intermediate points, the value of J will be a function, $J(v_1, \cdots, v_n)$, that will be analytic in $(v_1, \cdots, v_n)$ in the domain (Lemma §8)

$$A(u_i) \leqq v_i \leqq B(u_i) \qquad (i = 1, 2, \cdots, n). \tag{3}$$

The partial derivatives of the function J are given by the formulas

$$J_{v_i}(v_1, \cdots, v_n) = f_{v'}(u_i, v_i, p_i) - f_{v'}(u_i, v_i, q_i) \qquad (i = 1, 2, \cdots, n), \tag{4}$$

where p_i is the slope at (u_i, v_i) of the extremal joining (u_{i-1}, v_{i-1}) to (u_i, v_i), and q_i is the slope at (u_i, v_i) of the extremal joining (u_i, v_i) to (u_{i+1}, v_{i+1}). Since

$$f_{v'v'}(u, v, v') > 0$$

for every (u, v) in R and every v', the partial derivative (4) is zero, when and only when $p_i = q_i$. We have the result that *the function* $J(v_1, \cdots, v_n)$ *has a critical point* $(v_1, \cdots, v_n)$, *when and only when the points* (2) *all lie on a single analytic extremal.*

11. **Relations between critical points.** A lemma fundamental for our present purposes is derived from the Corollary to Theorem 8, page 392 of the paper of the author's already cited. Before stating the lemma let it be agreed that the positive normal to an analytic boundary of a region Σ will be understood to be that sensed normal that leads from points in Σ to points without Σ.

LEMMA 1. *Let there be given a closed region Σ in the space of the n variables $(x_1, \cdots, x_n)$, bounded by a closed analytic manifold, without singularity and homeomorphic with the interior and boundary of an $(n-1)$-dimensional hypersphere. In Σ let there be given a function $f(x_1, \cdots, x_n)$, of class C''' at each point of Σ, and possessing on the boundary of Σ a normal directional derivative that is positive. Suppose the critical points of $f(x_1, \cdots, x_n)$ are all of rank n. Of the type numbers k of these critical points, let m be the maximum. Let M_k be the number of critical points of type k. Then between the integers M_k the following relations hold:*

$$
\text{(R)}\quad
\begin{aligned}
&1 \leqq M_0,\\
&1 \geqq M_0 - M_1,\\
&1 \leqq M_0 - M_1 + M_2,\\
&1 \geqq M_0 - M_1 + M_2 - M_3,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&1 \leqq [M_0 - M_1 + M_2 - M_3 + - \cdots + (-1)^{m-1}M_{m-1}](-1)^{m-1},\\
&1 = M_0 - M_1 + M_2 - M_3 + - \cdots + (-1)^m M_m.
\end{aligned}
$$

In terms of the functions of the Lemma, §8, we can say that the domain of the points $(v_1, \cdots, v_n) = (V)$ for which the function $J(v_1, \cdots, v_n)$ of §10 is defined, is a rectangular hyperspace made up of points (V) satisfying the inequalities

$$
A(u_i) \leqq v_i \leqq B(u_i) \qquad (i = 1, 2, \cdots, n), \tag{1}
$$

where the numbers $(u_1, \cdots, u_n)$ are constants specified in §10. The boundary of the domain obviously consists of hyperrectangles on $2n$ hyperplanes

$$
v_i = A(u_i), \quad v_i = B(u_i) \qquad (i = 1, 2, \cdots, n),
$$

taken separately.

To apply Lemma 1 of this section we need to find the normal directional derivative of $J(v_1, \cdots, v_n)$ at points on the boundary of (1).

Let $(a_1, a_2, \cdots, a_n)$ be a point (V) of (1) for which $a_1 = A(u_1)$, that is, a point on the bounding hyperplane, $v_1 = A(u_1)$. If we hold the variables $(v_2, \cdots, v_n)$ constantly equal to $(a_2, \cdots, a_n)$, respectively, and let v_1 decrease so as to pass through $v_1 = a_1$, the corresponding point $(v_1, a_2, \cdots, a_n)$ will vary on a normal to the hyperplane $v_1 = A(u_1)$, and pass out of the domain (1). But the point $(v_1, \cdots, v_n) = (a_1, \cdots, a_n)$ will correspond (§10) to a succession of $n+1$ extremal segments joining A to B in R. Of these segments the first two will adjoin each other in R at the point $(u, v) = (u_1, a_1)$. The partial derivative of $J(v_1, \cdots, v_n)$ with respect to v_1 at the point $(a_1, \cdots, a_n)$ can be obtained from (4), §10, by putting $i = 1$. Using the law of the mean we then obtain the equation

$$
J_{v_1}(a_1, a_2, \cdots, a_n) = f_{v'v'}[u_1, a_1, \bar{p}][p_1 - q_1], \tag{2}
$$

where $\bar{p}$ is a number between p_1 and q_1. From the extremal-convex nature of the boundary it follows that $p_1 < q_1$. Thus this derivative (2) is negative.

The directional derivative of J along the normal to the hyperplane $v_1 = A(u_1)$ at the point $(a_1, a_2, \cdots, a_n)$ is to be taken in the sense that leads out of (1), that is in the sense of decreasing v_1. This directional derivative is thus equal to the left hand member of (2) multiplied by (-1), and is

accordingly positive. Similarly the directional derivative of J along a positive normal to any of the other hyperplanes bounding (1) may be seen to be positive.

But we cannot as yet apply the lemma on critical points because the boundary of (1) is made up of portions of $2n$ hyperplanes instead of one analytic manifold. To meet this difficulty let O be any interior point of (1), and let straight line rays be drawn from O to each point of the boundary of (1). It follows from the results of the preceding paragraph that the directional derivative of $J(v_1, \cdots, v_n)$, at a point P on the boundary of (1), taken along the ray joining O to P in the sense that leads from O to P is always positive. Now the domain (1) can be projectively transformed into an n-dimensional hypercube that lies in the space of $(y_1, \cdots, y_n)$, and that is bounded by the $(n-1)$-dimensional hyperplanes

$$y_i = \pm 1 \qquad (i = 1, 2, \cdots, n). \tag{3}$$

This hypercube can be approximated to by the analytic manifold

$$y_1^{2r} + y_2^{2r} + \cdots + y_n^{2r} = 1, \tag{4}$$

where r is a positive integer. In fact, if e be a positive constant, it is easy to show that for r sufficiently large, points on the above hypercube and the manifold (4) that lie on the same ray issuing from the origin will be within a distance e of one another.

Let now the hypercube be projectively transformed back into (1), and suppose the manifold (4) goes into a manifold M. Let the point O from which rays were drawn in (1) be the image of the origin in the hypercube. The rays in (1) will each meet the manifold M in a single point. If r be sufficiently large, M will approximate the domain (1) so closely that the directional derivatives of J at points of M on the rays issuing from O in the sense that leads away from O, will all be positive. It follows that the outer normal directional derivatives at points of M will be positive.

A second requirement on M is that it approximate the domain (1) so closely that it contain in its interior all the critical points of $J(v_1, \cdots, v_n)$ that (1) contains. The manifold M will serve as the manifold Σ of Lemma 1 of this section.

The critical points of Lemma 1 are of rank n. For the moment, then, we restrict ourselves to extremals joining A to B on which A is not conjugate to B. Reference to Theorem 2 of §6 and Lemma 1 of this section gives the lemma.

LEMMA 2. *Let there be given regions S and S_1, and integrand F, satisfying the hypotheses* I, II, III, *and* IV, *of* §7 *and* §8. *In S_1 let there be given two*

points A and B which are joined by no extremals on which A is conjugate to B. Let the number of extremals joining A to B in S_1 on which there are k conjugate points of A prior to B be denoted by M_k. Let m be an integer equal to the maximum of these integers k. Then between the numbers M_k the relations (R) *of Lemma* 1 *hold.*

12. **A theorem in the large.** We seek now to remove from Lemma 2, §11, the restriction that on no extremal joining A to B in S_1 is A conjugate to B. Before doing this it will be necessary to recall certain results obtained by Lindeberg.*

Let there be given in R an analytic extremal E_0 of the form

$$v = E(u), \qquad c \leqq u \leqq d,$$

joining A to B, and on which the point at which $u=c$ has for its $(m+1)$st conjugate point the point at which $u=d$. Let α be the slope at A of any extremal through A. In particular let $\alpha=\alpha_0$ be the slope of E_0 at A. That part of the envelope of the family of extremals through A which lies in the neighborhood of B, will not here consist merely of the point B. For according to the envelope theory this could only happen if all the extremals through A with slopes α near α_0 should pass through B, contrary to results in §9.

According to Lindeberg, the envelope T of the family of extremals passing through A, with a slope α near α_0, will then be of the form

$$\begin{aligned} v - E(u) &= (\alpha - \alpha_0)^{r+1}K(\alpha), \quad r > 0, \qquad K(\alpha_0) \neq 0, \\ u - d &= (\alpha - \alpha_0)^r H(\alpha), \qquad\qquad\quad H(\alpha_0) \neq 0, \end{aligned}$$

where $K(\alpha)$ and $H(\alpha)$ are analytic in α at $\alpha=\alpha_0$, and r is a positive integer.

Three classes of envelopes can be distinguished:

Class I $\quad r$ is odd;

Class II $\quad r$ is even and $H(\alpha_0)<0$;

Class III $\quad r$ is even and $H(\alpha_0)>0$.

Class I. Here the envelope T is tangent to the extremal E_0 at B, has no cusp there, and lies wholly on one side of E_0 near B. If the class of extremals through A be restricted to extremals E for which $|\alpha-\alpha_0|$ is sufficiently small, we can say that through each point P, not on T, but sufficiently near B, and on the same side of T as E_0, there will pass just two extremals of the set E. On these two extremals the type number k, that is, the number of points which are conjugate to A and prior to P, will equal m and $m+1$,

* Lindeberg, Mathematische Annalen, vol. 59 (1904), p. 321.

respectively. Through any point P, not on T, but sufficiently near B and on the opposite side of T from E_0, there will pass no extremals of the set E.

CLASS II. Here T is tangent to the extremal E_0 at B, but has a cusp there. On the envelope near B, $u<d$, except for the point B. The two branches of the cusp lie on opposite sides of E_0. A point P sufficiently near B, within the cusp, but not on T, can be joined to A by three extremals neighboring E_0. On these three extremals the type numbers k will have the values $m+1$, $m+1$, and m, respectively. Through any point P sufficiently near B, without the cusp, but not on T, there passes just one extremal issuing from A and neighboring E_0. On this extremal $k=m+1$.

CLASS III. Here T is tangent to the extremal E_0 at B, but has a cusp there. On the envelope near B, $u>d$, except for the point B. The two branches of the cusp lie on opposite sides of E_0. A point P, sufficiently near B, within the cusp, but not on T, can be joined to A by three extremals neighboring E_0. On these three extremals k has the values m, m, and $m+1$, respectively. Through each point P sufficiently near B, without the cusp, and not on T, there passes just one extremal issuing from A and neighboring E_0. On this extremal $k=m$.

In the following theorem there are a number of conventions to be adopted. Let g be an extremal joining A to B on which there are m points conjugate to A prior to B. If B is not conjugate to A, g is to be counted as one extremal of type $k=m$. *If B is conjugate to A, and g belongs to Class* I, *of the preceding classification, g is to be counted as two extremals, of types $k=m+1$ and $k=m$ respectively. If B is conjugate to A, and g belongs to Class* II, *or to Class* III, *then g is to be counted as one extremal of type $k=m+1$, or of type $k=m$, respectively.* We can now prove the following theorem "in the large."

THEOREM 3. *Let there be given regions S and S_1 and an integrand F satisfying hypotheses* I, II, III, *and* IV, *of* §7 *and* §8. *Let A and B be any two points of S_1. A first conclusion is that there are at most a finite number of extremals g joining A and B and lying in S_1. Let M_k be the number of these extremals g of type k, counted according to the conventions preceding this theorem, and let m be the maximum of these integers k. Then between the numbers M_k the relations* (R) *of Lemma* 1, §11, *hold true.*

That there are at most a finite number of extremals g joining A to B in S_1, was proved in §9. If on no extremal g, A is conjugate to B, the remainder of the theorem follows from Lemma 2, §11.

If there are a number of extremals g on which A is conjugate to B, each of these extremals g will be tangent at B to an envelope T of extremals issuing from A and neighboring that g. Let L be a short straight line segment

passing through B, lying in S_1, and tangent to none of these envelopes. Any point P on L, not B, but sufficiently near B, will lie on none of these envelopes. Concerning the possibility of joining A to P by extremals g' in S_1, we can state the following:

Corresponding to any extremal g on which A is not conjugate to B there will exist one extremal g' neighboring g, joining A to P, and of the same type as g.

If g be an extremal on which A is conjugate to B, and g is of Class II, the point P if sufficiently near B, will lie without the corresponding cusp. According to the preceding description of Class II there will then be just one extremal g' joining A to P and neighboring g. On g', A will not be conjugate to P, and the number of conjugate points of A prior to P will equal the type number k which we have agreed to assign to g. The facts are similar for extremals g of Class III.

If g be any extremal of Class I, joining A to B, let T be the corresponding envelope neighboring B. We consider two cases. In Case I we suppose P lies on the same side of T as g. In this case A and P can be joined by two extremals neighboring g, on which A is not conjugate to P, and whose type numbers are the numbers we have agreed to assign to g. In Case II we suppose P lies on the opposite side of T from g. In this case there will exist no extremals neighboring g' passing from A to P. Relative to this case we observe that if the relations (R) of Lemma 1, §11, hold between any given set of integers M_k, these relations will also hold if we replace M_i and M_{i-1} by M_i+1 and $M_{i-1}+1$, for any particular i $(i=1, 2, \cdots, n)$. Finally, if P be sufficiently near B, there will be no other extremals g' joining A to P than those just enumerated. For if there were more, we could prove by a limiting process that there would be other extremals g joining A to B besides those first supposed to exist.

The extremals which we have just proved join A and P are all extremals on which A is not conjugate to P. Concerning them Lemma 1 of §11 holds. If there are no extremals g joining A to B, of Class I, Case II, the theorem follows directly. But if there are extremals g of Class I, Case II, the relations of Lemma 1, §11, hold if all extremals g be counted except those of Class I, Case II. As we have already noted, the counting of extremals of Class I, Case II, as if they were in Class I, Case I, will not alter the validity of the relations (1) of §11, if these relations hold true prior to such a change. Thus the theorem is proved.

NOTE. *We could prove, by the methods of the paper on critical points, that extremals g of Class* I *could be omitted from the count entirely, and the relations* (R), §11, *still hold true. Such a proof would, however, lead us too far astray.*

Another important question is whether there are relations between the numbers M_k other than those affirmed in the Theorem. For the case of functions in general, apart from the calculus of variations, the answer is that there are no other relations without further hypotheses. More specifically the author has proved that *if there be given any set of integers M_i satisfying the relations* (R) *of Lemma* 1, §11, *there exists a function f of class C'' within and on a unit $(n-1)$-sphere, which on this $(n-1)$-sphere satisfies the boundary conditions of Lemma* 1, *and which possesses for each i, M_i critical points of type i, and no other critical points of any sort.**

For the case of the calculus of variations the author has shown that if under the hypotheses of Theorem 3 there is more than one extremal joining A and B, then there are at least two such extremals of minimum type. Except for this, Dr. Richmond, National Research Fellow at Harvard, has recently shown, *for the case where $m<3$ in the Theorem, that an example can be set up in the calculus of variations corresponding to any set of integers M_i satisfying the relations* (R).

13. **Existence of extremal-convex boundaries.** Let there be given in the (x, y) plane an open or closed curve γ, without multiple points, and made up of a finite number of arcs of class C'. If γ is closed, the points neighboring γ and not on γ make up two disconnected regions. If γ is open, the points neighboring γ, not on γ, and lying on short perpendiculars to the component arcs of γ, slightly extended at the vertices, again make up two distinct regions. *These two disconnected regions will be called the sides of γ.*

The curve γ will be said to be extremal-convex relative to one of its sides S, if the angles at its vertices on the side S are between 0 *and π, and if an extremal tangent to γ at any point P has no other point than P in common with γ or S in the neighborhood of P.*

Let g be any open or closed curve of class C' and without multiple points. A curve g' of class C' will be said to lie *arbitrarily near g, in position and direction*, if corresponding to an arbitrarily small positive constant e, a one-to-one continuous correspondence can be set up between g and g', in such a fashion that corresponding points are within a distance e of one another, and direction cosines of tangents at corresponding points differ by at most e. We can now prove the following lemma:

LEMMA 1. *Let g be any extremal segment satisfying the hypothesis in* §1, *and in addition derived from a reversible problem* (§7).

(A) *Then there can be found a curve g_1 of class C''', arbitrarily near to g*

* Morse, *The analysis and analysis situs of regular n-spreads in $(n+r)$-space*, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 813–817.

in position and direction, lying wholly on an arbitrary side of g, and such that g_1 will be extremal-convex relative to the side of g_1 that does not contain g.

(B) *If g contains no conjugate point to its end points, then in addition to g_1 there can also be found a second curve g_2 with the same properties as g_1, except that g_2 will be extremal-convex relative to the side of g_2 that contains g.*

To prove (A) we take the problem into the (u, v) plane as in §2, so that g becomes a segment of the u axis. The differential equation of the extremals near g can be put in the form

$$v'' = A(u, v, v') \tag{1}$$

where $A(u, v, v')$ is of class C''' for (u, v) neighboring g, and any number v' sufficiently small in absolute value. Alongside of (1) let us consider the differential equation

$$v'' = A(u, v, v') + Mv, \tag{2}$$

where M is a positive constant such that on g

$$M > A_v(u, 0, 0). \tag{3}$$

Now $v \equiv 0$ will represent a solution of (2), as well as of (1). The differential equation of the first variation, corresponding to a solution of (2), set up in particular for g, on which $v \equiv 0$, will be

$$w'' = A_{v'}(u, 0, 0)w' + [A_v(u, 0, 0) + M]w. \tag{4}$$

Let us compare (4), by Sturm's method, with

$$w'' = A_{v'}(u, 0, 0)w'. \tag{5}$$

Since (5) has a solution $w \equiv \text{constant} \neq 0$, (4) has no solution except $w \equiv 0$ which vanishes twice. Hence there is no conjugate point on g to either end point of g, relative to the differential equation (2).

Accordingly there exists a curve segment g_1, which represents a solution of (2), which lies arbitrarily near g, and on which $v > 0$. A comparison of the value of v'', say v_2'', given by (2) for a v and v' on g_1, with the value, say v_1'', of v'' given by (1) for the same v and v', shows that $v_2'' > v_1''$. The proof is similar for the side of g where $v < 0$.

To prove (B), compare (1) with

$$v'' = A(u, v, v') - ev^3, \tag{6}$$

where e is a positive constant. The equations of first variation corresponding to a solution of (6), or a solution of (1), respectively, are identical when set up for g, that is for $v \equiv 0$. Accordingly there is no conjugate point on g

to any point on g, when g is considered as a solution of (6). Hence a curve segment g_1 exists that represents a solution of (6), that lies arbitrarily near g, and on which $v>0$. Part (B) follows on comparing the v'' given by (6) with that given by (1) for the same v and v'. Lemma 1 leads to the following lemma.

LEMMA 2. *Let there be given in the region S of* §1 *a region S' bounded by a simple closed curve γ made up of a finite number of extremal segments g of* §1, *at least two in number, and making interior angles between* 0 *and* π. *Suppose further that the problem is reversible.*

(A) *Then each arc g may be replaced by an arc g_1 of class C''', arbitrarily near g_1, within S', and such that the set of arcs g_1 form a simple closed curve extremal-convex relative to its interior.*

(B) *If the end points of each arc g have no conjugate points on that arc g, each arc g may be replaced by an arc g_2 of class C''', arbitrarily near g_2, exterior to S', and such that the set of arcs g_2 form a simple closed curve extremal-convex relative to its interior.*

NOTE: The proof shows that when the integrand is analytic the preceding curves g_1 and g_2 may be taken as analytic.

14. **An example.** Let a surface of revolution be defined by revolving an analytic open or closed curve G, without singularities, about an axis lying in a plane with G_1, but not intersecting G. Let the surface be referred to parameters (u, v), of which u measures the angle through which G has been revolved from an initial position, and of which v represents arc lengths measured along G from a point chosen on G. The meridians $u=$constant are all geodesics. The parallels $v=v_0$ are geodesics, if at the point $v=v_0$ the tangent to G is parallel to the axis of revolution. These facts follow at once from the differential equations of the geodesics.

Let us represent the surface in the (u, v) plane. Let there be given in the (u, v) plane a rectangle S, bounded by any two curves $u=$constant, and by any two curves $v=$constant which represent geodesics. The rectangle S can be replaced by a region S_1, interior to the rectangle, but differing from the rectangle arbitrarily little, and bounded by a curve extremal-convex relative to its interior (Lemma 2, §13). The curves $u=$constant will form the field of extremals contemplated in §8, so that Theorem 3, §12, applies to S_1.

If the curves $v=$constant bounding the rectangle S represent parallels on which the distance to the axis has a proper or improper minimum relative to neighboring parallels, then these curves $v=$constant have no conjugate points on them, as can be readily proved. The curves $u=$constant never

have conjugate points on them. According to Lemma 2, §13, the rectangle S can in this case be replaced by a region S_1 *slightly larger* than S, bounded by a curve again extremal-convex relative to its interior. To S_1 Theorem 3, §12, will then apply. In particular the torus presents several different types of regions S_1 to which Theorem 3, §12, applies.

PART III. THE TYPE NUMBER OF A NON-DEGENERATE PERIODIC EXTREMAL

15. The function $J(v_1, \cdots, v_n)$. We start here with the same assumptions regarding the integrand $F(x, y, \dot{x}, \dot{y})$ as we made in §1. We suppose here, however, that we have given a periodic extremal g of length ω and of class C'''. We again suppose $F_1(x, y, \dot{x}, \dot{y}) > 0$ along g.

Let u be the arc length measured along g in the given sense. Let d be any positive constant less than the minimum distance between any two successive conjugate points on g. By an *admissible integer n, and constants $u_1, u_2, \cdots, u_n$,* will be meant an integer $n > 1$, and constants u_i increasing with their subscripts, all less than $u_1 + \omega$, and such that no one of the closed intervals on g consisting of points corresponding to a segment of the u axis bounded by successive points of the set $u_1, u_2, \cdots, u_n, u_1 + \omega$, exceeds d in length. Let $h_1, h_2, \cdots, h_n$ be n short arcs of class C''', crossing g respectively at $u_1, u_2, \cdots, u_n$, but not tangent to g, and let v_i be the arc length measured along h_i as in §1.

Let the point on h_i at the distance v_i from g be denoted by (u_i, v_i). If v_i be sufficiently small in absolute value, the successive points of the set

$$(u_1, v_1),\ (u_2, v_2),\ \cdots,\ (u_n, v_n),\ (u_1, v_1)$$

can be joined by unique extremals arbitrarily near segments of g between successive points on g at which u takes on respectively the values $u_1, u_2, \cdots, u_n, u_1 + \omega$. The value of the integral J taken along this succession of extremal segments will be again denoted by $J(v_1, \cdots, v_n)$.

16. The second partial derivatives of $J(v_1, \cdots, v_n)$. As in §2, g can be mapped onto the u axis in the (u, v) plane, with the additional fact that here the transformation from the (x, y) to the (u, v) plane can be taken as one in which x and y will be functions of u and v with a period ω in u. Points in the (u, v) plane whose u coördinates differ by ω will be termed *congruent.* The periodic extremal g will be represented in the (u, v) plane by any segment of the u axis of length ω. The function $f(u, v, v')$ derived from $F(x, y, \dot{x}, \dot{y})$ as in §3 will have a period ω in u, as will the coefficients of the J. D. E. corresponding to the extremal $v \equiv 0$. As in §2, so here, the transformation from the (x, y) to the (u, v) plane may be made to preserve distances along g and the arcs h_i, which become respectively in the (u, v) plane the curves

$v \equiv 0$ and $u = u_i$. Thus the function $J(v_1, \cdots, v_n)$, set up in the preceding section, will here equal the value of the integral in the (u, v) plane taken along extremal segments joining the successive points

$$(u_1, v_1), \ (u_2, v_2), \ \cdots, \ (u_n, v_n), \ (u_1 + \omega, v_1) \tag{1}$$

of the (u, v) plane.

$J(v_1, \cdots, v_n)$ will have a critical point when $(v_1, \cdots, v_n) = (0, \cdots, 0)$. To determine the nature of that critical point we shall examine the second partial derivatives of J.

In §4 the points (u_1, v_1) and (u_n, v_n) played a special rôle because they were adjacent to the end points of the given extremal. Here a set of formulas giving the second partial derivatives should still hold after a circular permutation of the points $(u_1, v_1), \cdots, (u_n, v_n)$. Such a circular permutation would be equivalent to advancing all the subscripts by the same integer, provided for any integer i we set

$$u_{i+n} = u_i + \omega, \qquad v_{i+n} = v_i. \tag{2}$$

With (2) understood we now repeat the definitions and assertions of §4 associated with (1), (2), (3), (4), (5) and (6) of §4.

The values of J_{v_i} are again given by (7) of §4 subject not to (8) of §4, but to the limitations (3) as follows:

$$i = 1, 2, \cdots, n, \qquad v_0 = v_n, \qquad v_{n+1} = v_1. \tag{3}$$

Instead of (9) in §4 we have here

$$J_{v_i v_j} \equiv 0, \quad i \neq j, \quad | \, i - j \, | \neq 1, \text{ or } n - 1, \tag{4}$$

subject also to (3). Equations (10), (11), and (12), of §4, hold here exactly as given in §4. From (7) of §4, as qualified by (3) above, we obtain finally

$$\begin{aligned} {}_0J_{v_n v_1} &= - R(u_n) w'_{n,n+1}(u_n), \\ {}_0J_{v_1 v_n} &= R(u_1) w'_{1,0}(u_1). \end{aligned} \tag{5}$$

The two partial derivatives of J just given in (5) were given in §4 by (9) and were there zero; *they are the only two second partial derivatives of J for which the formulas of this section differ from those of* §4.

17. **Periodic extremals classified: non-degenerate, simply-degenerate, and doubly-degenerate.** Before going further it is necessary to consider more in detail the J. D. E. as set up in §3 for the extremal $v \equiv 0$ in the (u, v) plane. We note here however that $P(u)$, $Q(u)$, and $R(u)$, have the period ω in u. In terms of the solutions of this J. D. E. we distinguish three different kinds of periodic extremals E.

I. *There are no solutions of the J. D. E. with the period* ω *other than* $w(u)\equiv 0$.

II. *There is a set of solutions with period* ω *of the form* $Cw(u)$, *where* $w(u)\not\equiv 0$, *and* C *is any constant, but no other solutions with period* ω.

III. *Every solution of the J. D. E. has the period* ω.

In these three cases we shall say, respectively, that the periodic extremal is I, *non-degenerate,* II, *simply-degenerate,* and III, *doubly-degenerate.*

Let $p(u)$ and $q(u)$ be two solutions of the J. D. E. that satisfy the initial conditions

$$(1)\qquad \begin{array}{ll} p(0) = 1, & q(0) = 0, \\ p'(0) = 0, & q'(0) = 1. \end{array}$$

Abel's integral for these solutions becomes

$$(2)\qquad R(u)\,[p(u)q'(u) - p'(u)q(u)] \equiv \text{constant}.$$

A substitution of $u=0$ and of $u=\omega$ in (2), gives the well known result

$$(3)\qquad p(\omega)q'(\omega) - p'(\omega)q(\omega) = 1.$$

We state without proof the following known result.

The given periodic extremal will be non-degenerate, simply-degenerate, or doubly-degenerate, according as the rank of the matrix

$$(4)\qquad \left\| \begin{array}{ll} p(\omega) - 1, & q(\omega) \\ p'(\omega) \quad , & q'(\omega) - 1 \end{array} \right\|$$

is 2, 1, *or* 0.

18. **Non-degenerate periodic extremal segments: convex, concave, or conjugate.** Let us represent solutions $w(u)$ of the J. D. E. as curves in the (u, w) plane. We will now prove the following lemma.

LEMMA 1. *If the given periodic extremal is non-degenerate, a necessary and sufficient condition that in the* (u, w) *plane every point* $(0, b)$ *be capable of being joined to its congruent point* (ω, b) *by a solution of the J. D. E., is that* $u=0$ *be not conjugate to* $u=\omega$.

Suppose a point $(0, b)$, not $(0, 0)$, can be joined to the point (ω, b) by a solution of the J. D. E. Such a solution will be of the form

$$(1)\qquad bp(u) + cq(u), \qquad b \neq 0,$$

where c is a suitably chosen constant. Since the solution passes through (ω, b) we have

$$(2)\qquad bp(\omega) + cq(\omega) = b.$$

Now if $q(\omega)$ were zero, from (2) it would follow that $p(\omega)=1$, whence the rank of the matrix (4), §17, would be less than 2, contrary to the fact that the J. D. E. has no periodic solutions other than $w(u)\equiv 0$. Thus the condition $q(\omega)\neq 0$ is a necessary consequence of the existence of solutions of the J. D. E. joining any point $(0, b)$ to its congruent point (ω, b).

To prove that if $q(\omega)\neq 0$, any point $(0, b)$ can be joined to (ω, b) by a solution of the J. D. E., say $w(u, b)$, we exhibit such a solution, namely,

$$w(u, b) = \frac{b}{q(\omega)}\begin{vmatrix} p(u) & , & q(u) \\ p(\omega)-1, & & q(\omega) \end{vmatrix}. \tag{3}$$

Thus the lemma is proved.

From equation (3) we obtain the following:

$$w_u(\omega, b) - w_u(0, b) = \frac{-b}{q(\omega)}\begin{vmatrix} p(\omega)-1, & q(\omega) \\ p'(\omega) \quad , & q'(\omega)-1 \end{vmatrix}. \tag{4}$$

Now the determinant in (4) is not zero if the given periodic extremal is non-degenerate. The usual methods of the calculus of variations serve with the aid of (4) to furnish a ready proof of the following lemma.*

Lemma 2. *If there be given a non-degenerate periodic extremal g on which the point $u=u_0$ is not conjugate to $u=u_0+\omega$, then in the (u, v) plane any point (u_0, a), $a\neq 0$, neighboring $(u_0, 0)$, can be joined to $(u_0+\omega, a)$ by an extremal segment g'. If congruent points be regarded as identical, these extremal segments g' will make an angle α with themselves at (u_0, a) measured on the side of g' towards g, which in magnitude will be either (Case I) always less than π, or else (Case II) always greater than π. If $u_0=0$, Cases I or II will occur according as the sign of*

$$M = -\frac{1}{q(\omega)}\begin{vmatrix} p(\omega)-1, & q(\omega) \\ p'(\omega) \quad , & q'(\omega)-1 \end{vmatrix} \tag{5}$$

is positive or negative.

The extremal segment g taken from $u=u_0$ to $u=u_0+\omega$, will be said to be convex or concave according as $\alpha<\pi$ or $\alpha>\pi$. We shall term M the test quotient.

In case a point $u=u_0$ on a non-degenerate periodic extremal g is conjugate to its congruent point, we shall say that the segment of g from $u=u_0$ to $u_0+\omega$ is a *conjugate segment.* It may happen in the case of a non-degenerate periodic extremal segment that every point is conjugate to its congruent point, as examples would show. In any case we see that a non-

* Compare Hadamard, *Leçons sur le Calcul des Variations*, Paris, 1910, pp. 434–435.

degenerate periodic extremal segment from $u=u_0$ to $u=u_0+\omega$, is either *convex, concave*, or *conjugate.*

19. **Conjugate points on simply-degenerate and doubly-degenerate periodic extremals.** We prove the following lemma.

LEMMA 1. *Let there be given a simply-degenerate, periodic extremal. Then if $w(u)$ is any one of the* 1-*parameter family of periodic solutions of the J. D. E. which is not identically zero, the only points which are conjugate to their congruent points are points at which $w(u)$ is zero.*

Suppose the lemma false. In particular suppose that $u=a$ is a point conjugate to $u=a+\omega$, while $w(a)\neq 0$.

Let $w_1(u)$ be a solution of the J. D. E. which vanishes at a, but is not identically zero. Since a is conjugate to $a+\omega$ we have

$$w_1(a+\omega) = w_1(a) = 0. \tag{1}$$

Abel's integral gives

$$R(u)\left[w(u)w_1'(u) - w'(u)w_1(u)\right] = \text{constant}. \tag{2}$$

Upon successively substituting $u=a$, and $u=a+\omega$ in this integral we obtain

$$w_1'(a+\omega) = w_1'(a). \tag{3}$$

Because of (1) and (3) we infer that $w_1(u)$ is periodic. Since $w_1(u)$ and $w(u)$ are linearly independent, every solution of the J. D. E. is periodic, contrary to the fact that we are dealing with the simply-degenerate case, and not the doubly-degenerate case. Thus the lemma is proved.

Concerning conjugate points *on doubly-degenerate periodic extremals, we say simply that every point is conjugate to its first congruent point.* It would be a mistake to believe that this property is characteristic of doubly-degenerate periodic extremals, for it may occur, in particular, in the case of a non-degenerate periodic extremal, as examples would show.

20. **The rank and form of the matrix of second partial derivatives of J.** We prove the following

THEOREM 4. *Corresponding to the given periodic extremal g of* §15, *the symmetric matrix* a *of elements*

$$a_{ij} = {}_0J_{v_iv_j}$$

is of rank n, $n-1$, or $n-2$, according as g is non-degenerate, simply-degenerate, or doubly-degenerate. If g is non-degenerate a *is always in normal form.**

* Bôcher, loc. cit., p. 59.

The case where g is non-degenerate. To prove the theorem in this case we turn again to the (u, w) plane of the J. D. E. of §18, and in that plane consider the points

$$(1) \qquad (u_0, c_0), \ (u_1, c_1), \ \cdots, \ (u_n, c_n),$$

of which (u_0, c_0) is supposed congruent to (u_n, c_n). We join the successive points of (1) by curve segments representing solutions of the J. D. E., and denote the resulting curve by λ. The curve λ will represent a periodic solution of the J. D. E., if and only if the slope of λ at (u_0, c_0) equals its slope at (u_n, c_n), and λ has no corners at the remaining points of (1). These conditions will be fulfilled if equations (5) of §5 are satisfied for $i=1, 2, \cdots, n$, and for c_0 and c_{n+1} respectively replaced by c_n and c_1. Let equations (5) of §5, so altered and understood, be denoted by (5a). The J. D. E. will have a periodic solution, not identically zero, if and only if equations (5a) are satisfied by a set of constants $(c_1, \cdots, c_n)$ not all zero. Such a periodic solution is possible if, and only if, the matrix of the coefficients of $(c_1, \cdots, c_n)$ in (5a) is of rank less than n. With the aid of the results of §16 this matrix is seen to be identical with the matrix $\mathbf{a}$ of the present theorem, provided the factor $R(u_i)$ be removed, for each i, from the ith row of $\mathbf{a}$. Thus the J. D. E. has no periodic solution except $w\equiv 0$, if and only if the rank of $\mathbf{a}$ is n.

That the matrix $\mathbf{a}$ is always arranged in normal form when g is a non-degenerate periodic extremal, can now be proved after the manner of proof that the matrix $\mathbf{a}$ of §5 is in normal form.

The case where g is simply-degenerate. According to the proof already given the rank of $\mathbf{a}$ here is less than n. It remains to prove that the rank of $\mathbf{a}$ is $n-1$.

Of the points $(u_1, 0), \cdots, (u_n, 0)$ at least one is not conjugate to its congruent point, for otherwise it would follow from the lemma of §19 that the set of points $(u_1, 0), \cdots, (u_n, 0)$ are mutually conjugate, contrary to the original choice of $(u_1, u_2, \cdots, u_n)$. We can then suppose, without loss of generality, that $u=u_n$ has been so chosen as not to be conjugate to $u=u_n-\omega=u_0$.

Now the minor A_{n-1}, obtained from $\mathbf{a}$ by striking out the last row and column, is one identical in form with the minor A_{n-1}, considered in §5. According to Corollary 1, §5, A_{n-1} will not be zero, since $(u_0, 0)$ is not conjugate to $(u_n, 0)$. Thus in the simply-degenerate case the rank of $\mathbf{a}$ is $n-1$.

The case where g is doubly-degenerate. As in the preceding case the rank of $\mathbf{a}$ is less than n. Further, every point on $v=0$ is here conjugate to its congruent point. In particular the point $(u_0, 0)$ is conjugate to its congruent point $(u_n, 0)$. It follows from Corollary 1, §5, that $A_{n-1}=0$. But if we understand that $u_i+\omega=u_{i+n}$ for each integer i, then by advancing the subscripts

we can bring it to pass that $(u_i, 0)$ is denoted by $(u_n, 0)$. In the matrix **a** the principal minor obtained by deleting the ith row and column becomes the principal minor A_{n-1} when $(u_i, 0)$ becomes $(u_n, 0)$. Thus all the principal $(n-1)$-rowed minors of **a** are zero. But according to Corollary 1, §5, A_{n-2} is not zero, since $(u_0, 0)$ is conjugate to $(u_n, 0)$ and cannot at the same time be conjugate to $(u_{n-1}, 0)$. Thus **a** is in this case of rank $n-2$, and the proof of the theorem is complete.

21. **The type number of a non-degenerate periodic extremal.** We prove the following

THEOREM 5. *If the periodic extremal g of* §15 *is non-degenerate, the type number k of the corresponding critical point of the function $J(v_1, \cdots, v_n)$ of* §15 *will be independent of the choice of n among admissible integers n, and of the points $(u_1, \cdots, u_n)$ on g among admissible points $(u_1, \cdots, u_n)$, and may be determined as follows. Setting $u_n-\omega=u_0$, let m be the number of conjugate points to $u=u_0$, preceding $u=u_n$. If $u=u_0$ is conjugate to $u=u_n$, $k=m+1$. If $u=u_0$ is not conjugate to $u=u_n$, then $k=m$, or $m+1$, according as the segment of g from $u=u_0$ to $u=u_n$ is convex or concave* (§18).

The number k will be called the type number of g.

As previously we concern ourselves here with the symmetric matrix **a** of which the elements are

$$a_{ij} = {}_0J_{v_iv_j}.$$

According to Theorem 4, §20, the matrix a in the case of a non-degenerate periodic extremal, is of rank n, and arranged in normal form. The type number k desired is then simply the number of changes in sign of the principal minors $A_0, A_1, \cdots, A_n$ (cf. §5, §6). To proceed further we agree again to set $u_{i+n}=u_i+\omega$, and $v_{i+n}=v_i$ for all integers i.

Now the matrix whose elements are those in A_{n-1} would be identical with the matrix a of §5 and §6, if in §5 and §6 we were dealing with the segment of g for which u lies between $u=u_0$ and $u=u_n$, and if we had chosen $n-1$ intermediate points $(u_1, \cdots, u_{n-1})$ instead of n. To proceed further we distinguish between two cases.

Case I. *The point $u=u_0$ is not conjugate to $u=u_n=u_0+\omega$.* In this case it follows from Theorem 2, §6, that the segment of g for which u lies between $u=u_0$ and $u=u_n$ is of type m, and that $A_{n-1}\neq 0$. Thus there are m changes of sign in the sequence $A_0, A_1, \cdots, A_{n-1}$. Hence in Case I the number k of the present theorem is m or $m+1$ according to whether or not A_{n-1} and A_n have the same sign.

Now consider the n equations (5) of §5, altered as follows. We here

replace c_0 by c_n, c_{n+1} by c_1, and the zero of the right hand member of the nth equation by

$$A_{n-1}/A_n, \tag{1}$$

while finally we here multiply the left hand member of the ith one of these equations $(i=1, 2, \cdots, n)$ by $R(u_i)$. We denote the resulting set of equations by (5b). Equations (5b) can be solved for $(c_1, \cdots, c_n)$, giving in particular for c_n a positive value

$$c_n = \frac{A_{n-1}^2}{A_n^2}.$$

The interpretation of this solution is that there is a solution of the J. D. E. which in the (u, w) plane passes through the points

$$(u_0, c_n),\ (u_1, c_1),\ (u_2, c_2), \cdots, (u_n, c_n),$$

whose slope at the point (u_n, c_n) minus its slope at (u_0, c_n) equals the fraction (1) divided by $R(u_n)$.

This difference of slopes, and hence the sign of (1), is readily seen to be positive or negative according as the segment of g taken from $u=u_0$ to $u=u_n$ is convex or concave in the sense of §18. Thus in Case I, k equals m or $m+1$ according as the segment of g from $u=u_0$ to $u=u_n$ is convex or concave.

Case II. *The point $u=u_0$ is conjugate to $u=u_n=u_0+\omega$.* In this case $A_{n-1}=0$, according to Corollary I, §5. Since $u=u_0$ is conjugate to $u=u_n$, it cannot be conjugate to $u=u_{n-1}$. If then Theorem 2, §6, be applied to the segment of g from $u=u_0$ to $u=u_{n-1}$, we find that that segment is of type m, since there are m points conjugate to u_0 preceding u_{n-1}. Hence there are m changes of sign in $A_0, A_1, \cdots, A_{n-2}$. But according to the theory of regularly arranged quadratic forms, when $A_{n-1}=0$, A_{n-2} and A_n have opposite signs. Thus there are $m+1$ changes of sign in $A_0, A_1, \cdots, A_n$. Hence $k=m+1$ in this case.

We can now prove that the type number k is independent of the choise of n, and $(U)=(u_1, u_2, \cdots, u_n)$, among admissible integers n and points (U).

The formulas of §16 and §4 show that the partial derivatives

$${}_0J_{v_i v_j}$$

are continuous functions of the position of admissible points (U). Now any admissible point (U) can be varied continuously through admissible points (U) into any other admissible point (U) for which n is the same. During such a variation the terms of the sequence

(2) $$A_0, A_1, \cdots, A_n$$

will vary continuously, and A_n will never be zero. We can now prove that there can be no variation in the total count of changes of sign in (2). This will certainly be true if no member of the sequence (2) is ever zero. If A_i becomes zero (where i cannot be zero or n) then A_{i-1} and A_{i+1} will have opposite signs at that stage of the variation.* Thus there can be no variation in the total count of changes of sign in (2), and the statement in italics is proved, except for possible changes of n.

That permissible changes in n do not alter the number k follows from the fact that, if we hold u_0 fast and introduce or remove any set of points u_i so as to have an admissible set u_i left, the number k, as determined under Cases I and II, will be the same before and after the change. Thus the theorem is completely proved.

PART IV. THE TYPE-NUMBER OF A DEGENERATE PERIODIC EXTREMAL

22. **Simply-degenerate, isolated, analytic, periodic, extremals.** By an *isolated* periodic extremal is understood a periodic extremal g, in the neighborhood of which there is no other periodic extremal with a length neighboring that of g. In this section and the two following we assume that *we are dealing with a periodic extremal g given in the region S of* §1. *Concerning $F(x, y, \dot{x}, \dot{y})$ we make the same assumptions as in* §1 *together with the assumption that $F(x, y, \dot{x}, \dot{y})$ be analytic for the values of $(x, y, \dot{x}, \dot{y})$ admitted. The extremal g is to be without multiple points, isolated, simply-degenerate, and analytic. Along g, $F_1(x, y, \dot{x}, \dot{y})$ is to be positive. We denote g's length by ω.*

The region neighboring g can be mapped conformally on the region neighboring the u axis in the (u, v) plane in such a fashion that g corresponds to any segment of the u axis of length ω, and so that congruent points (u, v) and $(u+\omega, v)$ correspond to the same point (x, y) in S, but that otherwise the transformation is one-to-one. As in §3 we can derive from $F(x, y, \dot{x}, \dot{y})$ an integrand $f(u, v, v')$ corresponding to which $v=0$ is an extremal. According to the lemma of §19 only a finite number of points of g are conjugate to their congruent points in the simply-degenerate case. Let $(u, v)=(0, 0)$ be one of the points of g not congruent to its conjugate point. As is well known, for a sufficiently small constant $a\neq 0$, any point $(0, a)$ can be joined to its congruent point (ω, a) by an extremal segment g' neighboring g. Further, if *congruent points be considered as identical these extremal segments g' make an angle with themselves at (ω, a) measured on the side of g' toward g which in magnitude will now be shown to be either*

* Bôcher, loc. cit., p. 147.

(I) *Less than* π *for all vertices* (ω, a) *on one side of* g, *and greater than* π *for* (ω, a) *on the other side of* g;

(II) *Less than* π *for all vertices* (ω, a) *on either side of* g; *or*

(III) *Greater than* π *for all vertices* (ω, a) *on either side of* g.

Further, whether I, II, *or* III *occurs depends only upon* g *and the choice of the point* $(u, v) = (0, 0)$.

The family of extremals passing through the point $(u, v) = (0, a)$ with a slope b can be represented in the form

$$v = A(u, a, b), \qquad a^2 + b^2 \leqq e, \tag{1}$$

where $A(u, a, b)$ is a real analytic function of (u, a, b) for u on any finite segment of the u axis and for a corresponding positive constant e sufficiently small, where further

$$a \equiv A(0, a, b), \tag{2}$$

$$b \equiv A_u(0, a, b). \tag{3}$$

The condition that an extremal (a, b) pass from a point $(0, a)$ to (ω, a) is that

$$A(\omega, a, b) - a = 0. \tag{4}$$

Now we have

$$A_b(\omega, 0, 0) \neq 0, \tag{5}$$

since $(0, 0)$ has been chosen so as not to be conjugate to $(\omega, 0)$. Hence (4) possesses a solution

$$b = \beta(a), \qquad \beta(0) = 0, \tag{6}$$

where $\beta(a)$ is analytic in a at $a = 0$. The slope of an extremal (a, b) given by (6) at (ω, a), minus its slope at $(0, a)$, is

$$A_u(\omega, a, \beta(a)) - \beta(a). \tag{7}$$

This function is not identically zero in a. For otherwise extremals (a, b) given by (6) would make up a continuous family of periodic extremals, a case which has been barred. Hence we can write

$$A_u(\omega, a, \beta(a)) - \beta(a) = \overline{A}(a) = a^r k(a), \qquad k(0) \neq 0, \quad r > 1, \tag{8}$$

where $k(a)$ is analytic in a at $a = 0$. That r is an integer exceeding one is seen from the fact that

$$\overline{A}'(0) = -\frac{1}{A_b(\omega, 0, 0)} \begin{vmatrix} A_a(\omega, 0, 0) - 1, & A_b(\omega, 0, 0) \\ A_{u.a}(\omega, 0, 0) \quad , & A_{u.b}(\omega, 0, 0) - 1 \end{vmatrix}. \tag{9}$$

In terms of $p(u)$ and $q(u)$ of §17 this gives

$$(10)\qquad \bar{A}'(0) = -\frac{1}{q(\omega)}\begin{vmatrix} p(\omega)-1, & q(\omega) \\ p'(\omega), & q'(\omega)-1 \end{vmatrix}.$$

Now the determinant in (10) is always zero if the periodic extremal is degenerate (§17). Thus here $\bar{A}'(0)=0$ and $r>0$. The statements in italics follow at once from (8).

23. **A simply-degenerate periodic extremal. Cases: concave-convex, concave, convex.** A simply-degenerate, isolated, analytic, periodic, extremal g, taken from $u=0$ to $u=\omega$ will be said to be *concave-convex, convex, or concave, according as* I, II, *or* III *of the preceding section holds.* We have previously given similar definitions for a non-degenerate periodic extremal (§18) involving the terms convex and concave. *The concave-convex case did not present itself in the non-degenerate case.* In the present case we state in terms of r and $k(0)$ in (8) of the preceding section that the segment of g from $u=0$ to $u=\omega$ is

(I) *Concave-convex when r is even,*

(II) *Convex when r is odd and $k(0)>0$,*

(III) *Concave when r is odd and $k(0)<0$.*

24. **The type number of a simply-degenerate periodic extremal.** To define the type number of g we are going to modify the integrand $f(u, v, v')$ in such a fashion that we shall no longer have to do with a degenerate periodic extremal. Let $h(u)$ be any real analytic function of u with a period ω, and μ be a parameter. An integrand of the form

$$(1)\qquad f(u, v, v') + \mu v h(u)$$

reduces for $\mu=0$ to the original integrand. We will now prove the following theorem.

THEOREM 6. *Concerning the simply-degenerate periodic extremal g of* §22, *on which* $(0, 0)$ *can and will be chosen so as not to be conjugate to* $(\omega, 0)$, *we can say the following. It is possible to choose a function $h(u)$ real and analytic in u with a period ω, and values of μ arbitrarily near $\mu=0$ in such a fashion, that if $f(u, v, v')$ be replaced by the integrand* (1) *then we have the following*:

(A) *If g, taken from $u=0$ to $u=\omega$, is convex-concave, the modified problem will possess no extremals neighboring g of period ω in u.*

(B_1) *If g, taken from $u=0$ to $u=\omega$, is convex or concave, the modified problem will possess just one extremal E neighboring g of period ω in u.*

(B_2) *The extremal E of* (B_1) *will be non-degenerate, and if there are on g just m points conjugate to $u=0$ and preceding $u=\omega$, the type number of E as determined in Theorem* 5, §21, *will be m or $m+1$ according as g, taken from $u=0$ to $u=\omega$, is convex or concave.*

The choice of $h(u)$. The extremals which correspond to the integrand (1) for values of μ neighboring $\mu=0$, and which lie in the neighborhood of the extremal $\mu=0$, can be represented in the form

$$v = B(u, a, b, \mu), \qquad a^2 + b^2 + \mu^2 \leqq e, \tag{2}$$

with

$$a \equiv B(0, a, b, \mu), \tag{3}$$

$$b \equiv B_u(0, a, b, \mu), \tag{4}$$

where $B(u, a, b, \mu)$ is analytic in its arguments, for u on any closed interval, and e a correspondingly sufficiently small positive constant.

We will show that we can choose $h(u)$ so that

$$B_\mu(\omega, 0, 0, 0) \neq 0, \tag{5}$$

$$B_{u\mu}(\omega, 0, 0, 0) \neq 0. \tag{6}$$

Differentiation of the Euler equation with respect to μ will show that the function $B_\mu(u, 0, 0, 0)$, set equal to $w(u)$, satisfies

$$Rw'' + R'w' + (Q' - P)w = h(u) \tag{7}$$

where $R(u)$, $P(u)$, and $Q(u)$ are the functions already used in the J. D. E. set up for the extremal $v\equiv 0$, when $\mu=0$.

It follows from (3) and (4) that we have initially

$$\begin{aligned} w(0) &= B_\mu(0, 0, 0, 0) = 0, \\ w'(0) &= B_{u\mu}(0, 0, 0, 0) = 0. \end{aligned} \tag{8}$$

If we make use of the functions $p(u)$ and $q(u)$ of §17, it is readily seen that this solution $w(u)$ of (7) is given by

$$w(u) = B_\mu(u, 0, 0, 0) = \int_0^u \frac{h(t)}{R(0)} \begin{vmatrix} p(t), & q(t) \\ p(u), & q(u) \end{vmatrix} dt.$$

In particular

$$B_\mu(\omega, 0, 0, 0) = \int_0^\omega \frac{h(t)}{R(0)} \begin{vmatrix} p(t), & q(t) \\ p(\omega), & q(\omega) \end{vmatrix} dt, \tag{9}$$

$$B_{\mu u}(\omega, 0, 0, 0) = \int_0^\omega \frac{h(t)}{R(0)} \begin{vmatrix} p(t), & q(t) \\ p'(\omega), & q'(\omega) \end{vmatrix} dt. \tag{10}$$

Now the coefficients of $h(t)$ in the integrands of (9) and (10) are respectively 0 and $R_{(0)}^{-1}$ for $t=\omega$. They are both positive for t in an interval

$$\omega - e < t < \omega, \tag{11}$$

if e be a sufficiently small positive constant. Let $h_1(u)$ be a function which is identically zero as t ranges from $t=0$ to $t=\omega$, except that in the interval (11), $v=h_1(u)$ shall be equal to the v ordinate on a semicircle with end points at $(\omega-e, 0)$ and $(\omega, 0)$ and on which $v>0$. If in (9) and (10), $h(t)$ be replaced by $h_1(t)$, the resulting integrals would both be positive. Another fact of importance is that if e in (11) be a sufficiently small positive constant the corresponding function $h_1(t)$ will not only make the integrals (9) and (10) positive but will cause the ratio of the integral (9) to the integral (10) to be arbitrarily small. Now $h_1(t)$ can be approximated by a number of terms of a Fourier series with period ω. The resulting function, which will serve as our definition of $h(t)$, can be taken as a function approximating $h_1(t)$ so closely that *for this choice of* $h(t)$ *in* (9) *and* (10) *the ratio of* $B_\mu(\omega, 0, 0, 0)$ *to* $B_{u\mu}(\omega, 0, 0, 0)$ *will be arbitrarily small, and both* $B_\mu(\omega, 0, 0, 0)$ *and* $B_{u\mu}(\omega, 0, 0, 0)$ *will be positive.*

We can now settle *the question of the existence of periodic extremals near the extremal* $v\equiv 0$ *corresponding to the integrand* (1) *when* μ *is near* $\mu=0$. The conditions for a periodic extremal neighboring $v\equiv 0$ for μ near $\mu=0$ are

$$B(\omega, a, b, \mu) - a = 0, \tag{12}$$

$$B_u(\omega, a, b, \mu) - b = 0. \tag{13}$$

Now equation (12) reduces for $\mu=0$ to equation (4) of §22. But (4) of §22 is satisfied by the function $b=\beta(a)$ of (6), §22. Hence (12) is satisfied by $b=\beta(a)$ and $\mu=0$. Further

$$B_b(\omega, 0, 0, 0) \neq 0, \tag{14}$$

since on the extremal $v\equiv 0$, the point (0, 0) was chosen not conjugate to $(\omega, 0)$. Hence (12) admits a solution of the form

$$b = b(a, \mu) = \beta(a) + \mu R(a, \mu), \tag{15}$$

where $R(a, \mu)$ is analytic in a and μ, at $a=0$ and $\mu=0$.

We are next concerned with solving (13) subject to (12), that is with solving

$$B_u[\omega, a, b(a, \mu), \mu] - b(a, \mu) = 0. \tag{16}$$

Using the fact that $b(a, 0)=\beta(a)$ we obtain the identity

$$B_u[\omega, a, b(a, 0), 0] - b(a, 0) \equiv A_u[\omega, a, \beta(a)] - \beta(a),$$

which becomes with the aid of (8), §22,

$$\equiv a^r k(a), \qquad k(0) \neq 0.$$

Hence (16) becomes

$$B_u[\omega, a, b(\alpha, \mu), \mu] - b(a, \mu) = a^r k(a) + \mu S(a, \mu) = 0, \tag{17}$$

where $S(a, \mu)$ is analytic at $a=0$ and $\mu=0$. We can solve (17) for μ if $S(0,0) \neq 0$. But $S(0, 0)$ is the partial derivative with respect to μ, at $(a, \mu) = (0, 0)$, of the left hand member of (17). Thus

$$S(0, 0) = [1 - B_{ub}(\omega, 0, 0, 0)] \frac{B_\mu(\omega, 0, 0, 0)}{B_b(\omega, 0, 0, 0)} + B_{\mu u}(\omega, 0, 0, 0). \tag{18}$$

Now as we have seen we can choose $h(t)$ so that $B_{\mu u}(\omega, 0, 0, 0)$ will be positive at the same time that the ratio of $B_\mu(\omega, 0, 0, 0)$ to $B_{\mu u}(\omega, 0, 0, 0)$ is arbitrarily small, so that the term $B_{\mu u}(\omega, 0, 0, 0)$ will dominate the sign in (18). Thus $S(0, 0) \neq 0$ for a proper choice of $h(t)$. Hence, (17) admits a solution of the form

$$\mu = \mu(a) = a^r d(a), \qquad d(0) \neq 0, \tag{19}$$

where $d(a)$ is analytic at $a=0$. The solution of (12) and (13) is now given by (19), taken with (15).

Final proof of (A). Now the integer r in (19) is the integer r in (8), §22. If the given periodic extremal is convex-concave, r is even. In this case let μ be chosen arbitrarily small, different from zero, and opposite in sign to $d(0)$ in (19). For this μ, (19) will admit no real solution a neighboring $a=0$, and corresponding to this μ the problem will possess no periodic extremal neighboring the extremal $v \equiv 0$.

Proof of (B_1). If the given periodic extremal is convex or concave, r is odd, and corresponding to any value of μ say $\mu_1 \neq 0$, neighboring $\mu=0$, (19) gives a value of a, say a_1, and (15) then a value of b, say b_1, corresponding to which the modified problem for which $\mu=\mu_1$ will possess a *single* periodic extremal E, with initial point $(0, a_1)$ and initial slope b_1.

Proof of (B_2). *The sign of the test quotient M of* (5), §18, *determined for E.* To show that E is non-degenerate it will be sufficient to prove that the test quotient M of Lemma 2, §18, evaluated for E, is not zero. According to the results of §18, E, taken from $u=0$ to $u=\omega$, will be convex or concave according as M is positive or negative. By thus determining the sign of M we intend to prove that the non-degenerate extremal E is convex or concave according as the given simply-degenerate extremal $v \equiv 0$ is convex or concave, taking both extremals from $u=0$ to $u=\omega$.

More generally the quotient M of §18 set up for any extremal with parameters (a, b, μ) in (2), will be denoted by $M(a, b, \mu)$. We have

$$\frac{-1}{B_b}\begin{vmatrix} B_a - 1, & B_b \\ B_{ua} \quad , & B_{ub} - 1 \end{vmatrix} = M(a, b, \mu), \tag{20}$$

where in each of the partial derivatives we set $u = \omega$, but where (a, b, μ) may take on any values neighboring (0, 0, 0). The form (20) shows that in terms of the function $b(a, \mu)$ of (15)

$$M[a, b(a, \mu), \mu] \equiv \frac{\partial}{\partial a}[B_u\{\omega, a, b(a, \mu), \mu\} - b(a, \mu)].$$

With the aid of (17) we therefore have

$$M[a, b(a, \mu), \mu] \equiv ra^{r-1}k(a) + a^r k'(a) + \mu S_a(a, \mu). \tag{21}$$

We shall evaluate (21) for values of μ given by (19), thereby evaluating $M(a, b, \mu)$ for values of (a, b, μ) which correspond to periodic extremals. Thus

$$M[a, b\{a, \mu(a)\}, \mu(a)] \equiv ra^{r-1}k(0) + \cdots, \qquad k(0) \neq 0, \tag{22}$$

where the terms omitted are of higher order than $r-1$ in a.

Now according to the results of §23 the given simply-degenerate extremal $v \equiv 0$, taken from $u = 0$ to $u = \omega$, is convex or concave when r is odd, and more particularly is convex or concave according as $k(0)$ is positive or negative. It follows from (22) and (19) that if E corresponds to a parameter $\mu \neq 0$, sufficiently small in absolute value, then E will be convex or concave according as the extremal $v \equiv 0$ is convex or concave (§ 18).

Finally the type of the non-degenerate extremal E, determined according to Theorem 5, §21, equals the number m of points on E conjugate to $u = 0$ and preceding $u = \omega$, plus one when E is concave, and exactly m when E is convex. But this number m will equal the corresponding number determined on the extremal $v = 0$, provided only that μ be sufficiently small in absolute value. Thus part (B_2) of the theorem is proved.

In accordance with the results of the preceding theorem, the given simply-degenerate periodic extremal g, in case it is convex or concave, will be said to be equivalent in type to the non-degenerate extremal E whose existence is affirmed by the preceding theorem, and in case it is convex-concave will be said to be equivalent to a null set of extremals and to be neutral in type.

25. **A doubly-degenerate, isolated, analytic, periodic, extremal.** We make the same assumptions here as we made in the case of a simply-degenerate periodic extremal (§22), except that here the given extremal g

is to be doubly-degenerate (§17). We will reduce the determination of the type of g to the cases already considered, that is, the non-degenerate and simply-degenerate cases.

THEOREM 7. *Concerning the preceding doubly-degenerate, isolated, periodic extremal g, and the corresponding simplified integrand $f(u, v, v')$ for which g becomes the extremal $v\equiv 0$, we can say the following. It is possible to choose a function $h(u)$ analytic and periodic in u with the period ω, such that the extremals corresponding to the modified integrand*

$$f(u, v, v') + \mu v h(u),$$

for properly chosen values of the parameter μ, neighboring $\mu = 0$, will include at most a finite set σ of extremals with the period ω in u that lie in the neighborhood of $v\equiv 0$, and such that none of these extremals will be doubly-degenerate.

To prove Theorem 7 we proceed as in the proof of Theorem 6, §24. We represent the extremals neighboring $v\equiv 0$ as in §24 and choose $h(u)$ in the same way, so that we may regard equations (1) to (13) of §24 as holding here. At this point the two proofs diverge, since (14) of §24 does not hold here. In terms of the function $A(u, a, b)$ of §22 of the unmodified problem, equations (12) and (13) of §24 become

$$B(\omega, a, b, \mu) - a \equiv A(\omega, a, b) - a + \mu D(a, b, \mu) = 0, \tag{1}$$

$$B_u(\omega, a, b, \mu) - b \equiv A_u(\omega, a, b) - b + \mu E(a, b, \mu) = 0, \tag{2}$$

where $D(a, b, \mu)$ and $E(a, b, \mu)$ are analytic in (a, b, μ) at $(a, b, \mu) = (0, 0, 0)$. With the aid of (5) and (6), §24, we see that

$$D(0, 0, 0) = B_\mu(\omega, 0, 0, 0) \neq 0, \tag{3}$$

$$E(0, 0, 0) = B_{u\mu}(\omega, 0, 0, 0) \neq 0. \tag{4}$$

Because of (3) and (4), (1) and (2) can be solved for μ. These solutions take the forms, respectively,

$$\mu = [A(\omega, a, b) - a]G(a, b), \tag{5}$$

$$\mu = [A_u(\omega, a, b) - b]H(a, b), \tag{6}$$

where $G(a, b)$ and $H(a, b)$ are analytic at $(a, b) = (0, 0)$ and do not vanish there. To solve (5) and (6) simultaneously we are led to the equation

$$[A(\omega, a, b) - a]G(a, b) = [A_u(\omega, a, b) - b]H(a, b). \tag{7}$$

We distinguish between two cases:

Case I: Equation (7) holds identically.

Case II: Equation (7) does not hold identically.

The proof in Case I. In this case neither of the differences

$$A(\omega, a, b) - a, \quad A_u(\omega, a, b) - b$$

can vanish along any real analytic arcs in the (a, b) plane neighboring $(a, b) = (0, 0)$. For if either of these two differences vanished along such real arcs, according to (7) the other difference would also so vanish, and the extremals for the unmodified problem $\mu = 0$ would include a family of periodic extremals contrary to the assumption that g is isolated. Except at $(0, 0)$, the two members of (7) must then be of one sign, say positive, throughout the neighborhood of $(a, b) = (0, 0)$. If then μ be chosen negative and sufficiently near $\mu = 0$, there will be no solution of (5) and (6) and hence no solution of (11) and (12). For each such choice of μ the modified problem will possess no periodic extremals neighboring g.

Proof in Case II. It may happen in this case that (7) possesses no real solutions neighboring $(a, b) = (0, 0)$ other than $(0, 0)$. If this occurs, then for any choice of $\mu \neq 0$, sufficiently near $\mu = 0$, (1) and (2) admit no real solutions (a, b, μ) and the modified problem possesses no periodic extremals neighboring g.

If, on the other hand, (7) possesses real solutions $(a, b) \neq (0, 0)$, arbitrarily near $(a, b) = (0, 0)$, these real solutions will make up a finite number of real analytic arcs representable in a one-to-one manner by function pairs of the form

$$\begin{aligned} a &= a(t), \qquad a(0) = 0, \\ b &= b(t), \qquad b(0) = 0, \end{aligned} \tag{8}$$

where $a(t)$ and $b(t)$ are real analytic functions of t at $t = 0$, and are not both identically zero. The differences

$$\begin{aligned} &A\,[\omega, a(t), b(t)] - a(t), \\ &A_u[\omega, a(t), b(t)] - b(t) \end{aligned} \tag{9}$$

cannot both be identically zero, for otherwise $a(t)$, $b(t)$ would correspond in the unmodified problem to a continuous family of periodic extremals. On the other hand neither of the functions in (9) can be identically zero in t without the other being identically zero in t, as follows from (7). We can then set

$$A[\omega, a(t), b(t)] - a(t) \equiv t^s F(t), \qquad F(0) \neq 0, \qquad s > 0, \tag{10}$$

where $F(t)$ is analytic at $t = 0$. From (5) it follows that the values of μ that go with (8) to give a solution of (1) and (2) are representable in the form

$$\mu = \mu(t) \equiv t^s K(t), \qquad K(0) \neq 0, \tag{11}$$

where $K(t)$ is analytic at $t=0$. For any particular t, neighboring $t=0$, and corresponding values of $a(t)$, $b(t)$, and $\mu(t)$, the corresponding extremal g' will be periodic.

Among the conditions that g' be doubly-degenerate are that for the value of t that gives g' (§17),

$$B_a[\omega, a(t), b(t), \mu(t)] = 1, \tag{12}$$

$$B_b[\omega, a(t), b(t), \mu(t)] = 0. \tag{13}$$

We proceed to show that (12) and (13) do not hold identically in t. For if (12) and (13) did hold identically in t, a performance of the following indicated differentiation would show that

$$\frac{\partial}{\partial t}\{B[\omega, a(t), b(t), \mu] - a(t)\} \equiv 0, \quad \mu = \mu(t), \tag{14}$$

was an identity in t, where, as indicated, μ is to be held fast during the differentiation and set equal to $\mu(t)$ thereafter. If use be made of (1) and (10) we obtain the identity

$$B[\omega, a(t), b(t), \mu] - a(t) \equiv t^sF(t) + \mu D[a(t), b(t), \mu], \tag{15}$$

and the partial derivative (14) becomes, upon using (11) and (15),

$$st^{s-1} F(t) + t^s F'(t) + t^s K(t)[D_a \cdot a'(t) + D_b \cdot b'(t)]. \tag{16}$$

This function does not vanish identically since $s>0$ and $F(0)\neq 0$. Hence (12) and (13) do not hold identically, and accordingly hold simultaneously for no value of t, neighboring $t=0$, other than $t=0$.

Now to any real value of μ not zero, but sufficiently near zero, there will correspond, under (11), either no real value, one real value, or two real values of t, according to the evenness, or oddness of s, and the sign of $K(0)$. To such a μ there will then correspond, by virtue of (8), no periodic extremal, one periodic extremal, or two periodic extremals, and none of these extremals will be doubly-degenerate, since (12) and (13) will not both hold.

Similarly there may arise a finite number of other periodic extremals from real solutions of (7) other than $a(t)$, $b(t)$, but in any case for a $\mu\neq 0$, and sufficiently near $\mu=0$, these periodic extremals will not be doubly-degenerate. Thus the theorem is proved.

Let the set σ of periodic extremals appearing in Theorem 7 be modified by replacing each simply-degenerate periodic extremal g_1 of σ by an equivalent non-degenerate extremal, or a null set of extremals, according to the conventions at the end of §24. *The set σ_1 of non-degenerate extremals thereby obtained will be*

said to be equivalent in type to the given doubly-degenerate extremal g. If neither σ nor σ_1 contains any extremals, g will be said to be equivalent to a null set of extremals, and to be of neutral type.

PART V. RELATIONS IN THE LARGE BETWEEN PERIODIC EXTREMALS

26. The integrand and regions S and S_1. We make here again the assumptions I and II of §7 and §8, qualifying F and S. We replace III by III′, and IV by IV′ as follows:

III′. *Let there be given a closed region S_1 consisting of points interior to S bounded by two simple closed curves β_1 and β_2, of which β_2 lies within β_1. We suppose both β_1 and β_2 consist of a finite number of analytic arcs without singularities and are extremal-convex relative to S_1 in the sense of* III §8.

IV′. *We assume that we have in S a proper field of extremals representable in the form*

$$x = h(u, v), \qquad y = k(u, v), \tag{1}$$

where u is the parameter and v is the arc length measured along the extremals, and $h(u, v)$ and $k(u, v)$ have a period ω in u. We assume further that for any constant a and interval

$$a \leqq u < a + \omega, \tag{2}$$

the field (1) *covers S_1 in a one-to-one manner, and that at each point (u, v) that corresponds to a point (x, y) in S_1 the functions* (1) *are single-valued and analytic in u and v, and*

$$\begin{vmatrix} h_u, & h_v \\ k_u, & k_v \end{vmatrix} \neq 0.$$

The lemma of §8 will here be replaced by the following lemma of which the method of proof is very similar to that used in §8. No proof need be given.

LEMMA 1. *The region S_1 of the (x, y) plane will correspond under* (1) *to a region R of the (u, v) plane bounded by two unending arcs of the form*

$$\begin{aligned} &v = A(u), \quad A(u + \omega) = A(u), \\ &v = B(u), \quad B(u + \omega) = B(u), \quad A(u) < B(u), \end{aligned} \tag{3}$$

where $A(u)$ and $B(u)$ are of class C for all values of u, and analytic in u except for a finite set of values of u, on any interval of the form (2). *The interior points of R are the points (u, v) such that*

$$A(u) < v < B(u). \tag{4}$$

The correspondence between S_1 and any set of points of R limited as in (2) *will be one-to-one.*

As in §9, so here, it follows that all extremals, except the extremals u = constant, are representable in the form

$$v = M(u), \tag{5}$$

where $M(u)$ is an analytic function of u for all values of u that give points (u, v) in R. Further any extremal joining two points in R will have no points on the boundary of R, with the possible exception of its end points.

We shall concern ourselves at first with periodic extremals continuously deformable in S_1, in the (x, y) plane, into either one of the two closed boundary curves of S_1, taken just once. In R, in the (u, v) plane, these extremals will be representable in the form (5), with $M(u)$ possessing the period ω. A point of difference between the developments for periodic extremals and those for extremals joining two fixed points A and B, is that in the latter case we were able to prove, under hypotheses I, II, III, and IV, of §7 and §8, that there were at most a finite number of extremals joining A to B, while in the case of periodic extremals the hypotheses I, II, III′, and IV′ are not sufficient to bar the existence of analytic families of periodic extremals lying in S_1 and deformable into a boundary of S_1. Simple examples can be given to show the truth of this statement. However, cases where families of periodic extremals exist are certainly specialized in that the differential equation of the first variation, that is the J. D. E., set up for each member of such families, must possess a periodic solution not identically zero. Not only is this true but the differential equations of the variations of orders higher than the first must all possess periodic solutions. In excluding such families we are therefore excluding exceptional cases. We are the more justified in such exclusion by the fact that we are developing a theory that will serve to prove the existence of additional periodic extremals when a finite set of such extremals is given. If infinite sets of mutually deformable periodic extremals existed, then more than we hoped to prove would be granted. We state here the fundamental lemma which replaces Lemma 2 of §11.

LEMMA 2. *Let there be given regions S and S_1 satisfying* I *and* II *of* §7, *and* III′ *and* IV′ *of this section. In S_1 suppose there are at most a finite number of periodic extremals continuously deformable into either boundary of S_1, taken just once, and that all of these extremals are non-degenerate* (§17). *Let the number of these extremals which are of type k*(§21) *be denoted by M_k. Let m be the maximum of the integers k. Then between these numbers M_k the relations* (R) *of* §11 *hold.*

This lemma is proved with the aid of the fundamental lemma on critical points, §11, and Theorem 5, §21, in a manner similar to the manner of proof of Lemma 2, §11.

27. **The theorem in the large.** In order to remove the restriction from the above lemma that the periodic extremals appearing there be nondegenerate we cannot rely on the envelope theory as in the case of extremals joining two fixed points. The following lemma will serve our purpose. In this lemma let the $(n-1)$-dimensional hypersphere with radius r and center (A) in the space of $(v_1, \cdots, v_n)$ be denoted by $S^{A,r}$.

LEMMA 1. *Let there be given a function $J(v_1, \cdots, v_n)$ of class C''' at each point $(V)=(v_1, \cdots, v_n)$ of an open n-dimensional region Σ. Let $(A)=(a_1, \cdots, a_n)$ be a point of Σ at which $J(v_1, \cdots, v_n)$ has an isolated critical point. Let $J(v_1, \cdots, v_n, \mu)$ be a function of class C''' for (V) in Σ, and μ in the neighborhood of $\mu=0$, and such that in Σ*

$$J(v_1, \cdots, v_n) \equiv J(v_1, \cdots, v_n, 0). \tag{1}$$

Let e be a positive constant so small that $J(v_1, \cdots, v_n)$ has no critical point other than (A) within or on $S^{A,2e}$. Then for any fixed value of $\mu \neq 0$ and sufficiently small in absolute value, it is possible to replace $J(v_1, \cdots, v_n)$ by a function $\phi(v_1, \cdots, v_n)$ of class C''' throughout Σ, and of such sort that in $S^{A,e}$

$$\phi(v_1, \cdots, v_n) \equiv J(v_1, \cdots, v_n, \mu), \tag{2}$$

while in Σ but outside of $S^{A,2e}$

$$\phi(v_1, \cdots, v_n) \equiv J(v_1, \cdots, v_n), \tag{3}$$

and finally in the closed domain between $S^{A,2e}$ and $S^{A,e}$, $\phi(v_1, \cdots, v_n)$ has no critical points.

To prove this lemma we are going to use a function $h(x)$ that is of class C''' for all values of x, and is such that for the preceding constant e

$$\begin{aligned} h(x) &\equiv 1, \qquad |x| \leqq e, \\ h(x) &\equiv 0, \qquad |x| \geqq 2e. \end{aligned} \tag{4}$$

Such a function as $h(x)$ can readily be set up in terms of the elementary functions.

For points $(v_1, \cdots, v_n)$ of the closed domain between $S^{A,2e}$ and $S^{A,e}$ set

$$(5)\qquad D(v_1, \cdots, v_n, \mu) \equiv J(v_1, \cdots, v_n, \mu) - J(v_1, \cdots, v_n),$$

and now define ϕ for the same points $(v_1, \cdots, v_n)$ as follows:

$$(6)\ \phi(v_1, \cdots, v_n) \equiv J(v_1, \cdots, v_n) + D(v_1, \cdots, v_n, \mu)h([v_1^2 + \cdots + v_n^2]^{1/2}).$$

Now the constant μ can be chosen so small in absolute value that the function D of (6), as well as each of the partial derivatives D_{v_i}, is less in absolute value than a preassigned positive constant d throughout the whole region between $S^{A,2e}$ and $S^{A,e}$. For the same domain the sum of the squares of the first partial derivatives of $J(v_1, \cdots, v_n)$ exceeds some positive constant. It follows readily from (6), that for a $\mu \neq 0$, sufficiently small in absolute value, the function ϕ of (6) will have no critical point between $S^{A,2e}$ and $S^{A,e}$. We accordingly understand such a value of μ chosen, and hereafter held fast.

If now the function $\phi(v_1, \cdots, v_n)$ be defined at the remaining points of Σ by (2) and (3), it is readily seen that $\phi(v_1, \cdots, v_n)$ has the properties affirmed in the lemma.

We are now in a position to state the following theorem.

THEOREM 8. *Let there be given regions S and S_1 and integrand $F(x, y, \dot{x}, \dot{y})$ satisfying* I, II, II′, IV′ *of* §7 *and* §26. *In S_1 suppose there are at most a finite set T, of periodic extremals g, continuously deformable into a boundary curve of S_1. In the set T let each degenerate extremal be replaced by an equivalent set of non-degenerate extremals in accordance with the conventions at the ends of* §24 *and* §25. *Let M_k be the number of periodic extremals of type k in the resulting set, and m be the maximum of the numbers k. Between the numbers M_k the relations* (R) *of* §11 *hold.*

If the above periodic extremals g are all non-degenerate, the present theorem is identical with Lemma 2, §26.

Suppose on the other hand that not all of the above periodic extremals g are non-degenerate. To be specific, suppose the above set includes a simply-degenerate periodic extremal g_1 which taken from $u=0$ to $u=\omega$ is convex or concave. Then according to Theorem 6 of §24 it will be possible to modify the integrand by the introduction of a parameter μ in such a fashion, that for a suitably chosen value of μ, say μ_1, arbitrarily small in absolute value, there will appear, instead of g_1, in the neighborhood of g_1, a non-degenerate extremal E of what we have agreed to take as the type equivalent to g_1. Corresponding to this introduction of a parameter μ into the integrand, the function $J(v_1, \cdots, v_n)$ whose critical points correspond to the given periodic extremals will be replaced by a function $J(v_1, \cdots,$

$v_n, \mu)$. In setting up $J(v_1, \cdots, v_n, \mu)$ we can and will use the same field of extremals (IV′, §26) and corresponding parametric system (u, v) as in setting up $J(v_1, \cdots, v_n)$. The broken extremals along which $J(v_1, \cdots, v_n, \mu_1)$ is to be the integral will, however, be taken as the extremals corresponding to the integrand in which $\mu = \mu_1$. Let $(A) = (a_1, \cdots, a_n)$ be the point at which the unmodified function $J(v_1, \cdots, v_n)$ has the isolated critical point corresponding to g_1. By virtue of our modification of the integrand of the problem, $J(v_1, \cdots, v_n, \mu_1)$ will have, in the neighborhood of the point (A), a critical point (B) of the type termed equivalent to that of g_1.

It follows from the lemma of this section that we can set up a function $\phi(v_1, \cdots, v_n)$ that will have in a suitably chosen neighborhood of (A) no other critical point than (B), and that will be identical with $J(v_1, \cdots, v_n)$ without this neighborhood of (A).

We can similarly modify $J(v_1, \cdots, v_n)$ in the neighborhood of each other critical point (A) that corresponds to a simply-degenerate periodic extremal g_1, so that the modified function has in the neighborhood of (A) a critical point (B) of a type equivalent to that of g_1, in case g_1 is convex or concave, or has no critical point at all in the neighborhood of (A) in case g_1 is convex-concave, while except for the neighborhoods of these points (A), the modified function is identical with the original function $J(v_1, \cdots, v_n)$.

In case the set T includes doubly-degenerate periodic extremals, it follows from Theorem 7, §25, and Lemma 1 of this section, that we can first modify $J(v_1, \cdots, v_n)$ in such a fashion that the resulting function no longer has critical points corresponding to doubly-degenerate extremals. We can then, by additional successive modifications, obtain finally a function replacing $J(v_1, \cdots, v_n)$ whose critical points all correspond to non-degenerate extremals, including thereby all of the non-degenerate extremals of the original set T, together with the complete set of non-degenerate extremals equivalent to the degenerate extremals of T. The theorem then follows upon applying the lemma on critical points of §11 to the function finally evolved from $J(v_1, \cdots, v_n)$.

Part VI. Deformation theory

28. **Families of curves joining A to B.** In this part of the paper we shall show how the type of a given extremal segment can be characterized in terms of the possibility or impossibility of making certain deformations of families of curves joining A to B. The methods will be extended to the case of closed extremals in a later section. The results obtained, it is hoped by

the author, will serve as a part of a necessary basis for a "theory in the large" more extensive than any already developed.

By an m-family of curves, Z_m, will be understood a set of ordinary curves (B, p. 192) lying in the (x, y) plane and passing from an initial point A to a final point B; where, further, the set of all points of Z_m make up a single-valued continuous point function of the points on a product complex C_{m+1} obtained by combining an arbitrary point t on a closed interval of the t axis with an arbitrary point P on some m-dimensional manifold M_m; and where, finally, the dependence of the points of Z_m upon the points of C_{m+1} is such that to hold P fast and vary t gives a representation of the individual curves of Z_m by virtue of which they may be termed ordinary.*

The point P will be called the *parametric* point and the manifold M_m the *parametric* manifold.

Let there be given two families Z_m' and Z_m'' consisting of curves which join the same two points A and B. If the parametric manifolds of Z_m' and Z_m'' are *homeomorphic* the two m-families can be represented by the aid of the parametric manifold of either Z_m' or Z_m'', in particular, say, by the manifold M_m. *In such a case Z_m' and Z_m'' will be said to be mutually deformable if corresponding to each value of a parameter μ on the interval*

$$0 \leqq \mu \leqq 1$$

there exists an m-family $Z_m(\mu)$ of which $Z_m(0)$ is Z_m' and $Z_m(1)$ is Z_m'', while each m-family $Z_m(\mu)$ is representable in terms of the same parametric manifold M_m, and the same interval for t, and joins the same two points, A and B; and if further the complete set of points (x, y) on these m-families, $Z_m(\mu)$, by virtue of their dependence upon μ, t and P, make up a single-valued continuous point function of an arbitrary point on the product complex C_{m+2} obtained by combining an arbitrary point μ on its interval, an arbitrary point t on its interval, and an arbitrary point P on M_m.

In a deformation D such as the one just defined Z_m' will be called the *initial* family and Z_m'' the *final* family. The m-families $Z_m(\mu)$, for values of μ between 0 and 1, will be called the *intermediate families*. A point $P=P_0$ on M_m, and a value $\mu=\mu_0$ held fast while t varies, determine a curve in $Z_m(\mu_0)$. The curve $P=P_0$ of Z_m' will be said to be *replaced* in the deformation D when $\mu=\mu_0$, by the curve $P=P_0$ of $Z_m(\mu_0)$. Points on a curve P of Z_m', and points on any curve replacing P under D will be said to *correspond* if they are given by the same values of t. If we should hold P and t fast

* Cf. Veblen, The Cambridge Colloquium, 1916, Part II, *Analysis Situs*, p. 88.

in $Z_m(\mu)$, and vary μ, the resulting set of points would be the locus of points *corresponding* under D to a single point of Z_m'.

We hereby understand that all of the definitions of this section have been given in terms of (u, v) as well as of (x, y).

29. **Hypotheses. Fundamental lemmas on deformations.** We make here concerning $F(x, y, \dot{x}, \dot{y})$, and the given extremal g, the same hypothesis as in §1, except that here we suppose that F and g are of class C'''' instead of class C''', and that $F_1(x, y, \dot{x}, \dot{y})$ is positive, not only along g, but also for (x, y) on g and for $(\dot{x}, \dot{y})$ any two numbers not both zero. A consequence of the assumption that F be of class C'''' instead of class C''', is that the function $J(v_1, \cdots, v_n)$ set up in §1 is here of class C''' for $(v_1, \cdots, v_n)$ in the neighborhood of $(0, \cdots, 0)$. As in §2 we transfer the problem to the (u, v) plane, carrying g into a segment γ of the u axis.

By a *canonical* curve will be understood a succession of extremal segments joining the successive points of the set

$$(u_0, 0)(u_1, v_1), \cdots, (u_n, v_n)(u_{n+1}, 0), \tag{1}$$

determined as in §1. An m-family of canonical curves will be called a *canonical m-family.*

Let z stand for any positive constant a, b, c, d, e, etc. Let R_z denote the set of points (u, v) within a distance z of γ.

LEMMA 1. *Let R_1 be any region in the (u, v) plane enclosing γ in its interior and in which the problem is "regular." If a be a sufficiently small positive constant, the region R_a, consisting of the points (u, v) within a distance a of γ, will possess the following property. Any m-family Z_m consisting of curves that lie in R_a, join γ's end points, and give to J a value such that*

$$J \leqq J_0 - e^2, \tag{2}$$

where e is a positive constant, and J_0 is the value of J along γ, can be deformed within R_1, through the mediation of curves that always satisfy (2), *into an m-family of canonical curves. This deformation can be so made that if any of the curves of Z_m are canonical curves they are replaced in the deformation only by curves along which v is a function of u of at least class C.*

Before coming to the proof proper we make a number of preliminary statements and definitions.

(A) We can and will choose a positive constant r so small that of the intervals I_i:

$$u_i - r \leqq u \leqq u_i + r \qquad (i = 0, 1, \cdots, n+1), \tag{3}$$

no two successive intervals I_i and I_{i+1} have any points in common or contain any conjugate points of each other.

(B) If in the results obtained by Lindeberg* in a paper cited below we set

$$G(x, y, x', y') = (x'^2 + y'^2)^{1/2},$$

we readily obtain the following. Corresponding to the positive constant r just chosen in (A) there can be found a region R_b so small that if $\bar{\gamma}$ be any "ordinary curve" (B, p. 192) joining γ's end points within R_b and satisfying (2), if u and $\bar{u}$ are u coördinates of points $\overline{Q}$ and Q on $\bar{\gamma}$ and γ respectively, and if $\overline{Q}$ and Q lie at the same distance measured along $\bar{\gamma}$ and γ respectively from $(u_0, 0)$ or from $(u_{n+1}, 0)$, then

$$|\bar{u} - u| \leqq r.$$

(C) We will now choose a region R_c as follows. We first require the region R_c to lie within the region R_b of the preceding paragraph (B). Further we can and will choose c so small (B, pp. 275 and 307) that any two *distinct* points Q and Q' which both lie in one of the regions

$$u_{i-1} - r \leqq u \leqq u_i + r, \qquad v \leqq c \quad (i = 1, 2, \cdots, n+1) \tag{4}$$

can be joined in the order Q, Q' by an extremal segment E with the following properties:

(a) The extremal E gives a minimum to J relative to all "ordinary curves" joining Q to Q' and lying in (4).

(b) The coördinates (u, v) of points of E are functions of at least class C' of the coördinates of Q and Q' and of the distance s of the points (u, v) from Q measured along E.

(c) In case Q and Q' lie in two successive intervals I_i and I_{i+1} of (A), the coördinate v of a point (u, v) on E will also be a function of class C' of u as well as of the coördinates of Q and Q'.

(D) Finally we choose a region R_d which we will prove can serve as the region R_a of the lemma. We first require that d be so small that if the end points of any of the extremal segments E described in (C) lie in R_d the whole of E will lie in R_c. A second requirement on d will be added later.

Now let Z_m be any m-family each curve of which lies within R_d and satisfies the relation (2). In accordance with the conventions of §28, let P be the parametric point of the family Z_m, and M_m the parametric manifold on which P lies. We come now to the deformation of Z_m into an m-family of canonical curves. This deformation will be given as the resultant of two deformations,

* J. W. Lindeberg, *Über einige Fragen der Variationsrechnung*, Mathematische Annalen, vol. 67 (1909), p. 351, §8.

D' and D'', which obviously can be combined into a single deformation if the final m-family of D' is identical with the initial m-family of D''.

Let each curve of Z_m be divided into $n+1$ successive segments of which the ith, say g_i, consists of points for which the arc length s measured from $(u_0, 0)$ satisfies

$$u_{i-1} \leqq s \leqq u_i \qquad (i = 1, 2, \cdots, n),$$

and when $i=n+1$ consists of the remaining points on the given curve. In accordance with the preceding results, (A) and (B), the coördinates of the points of g_i will satisfy

$$u_{i-1} - r \leqq u \leqq u_i + r \qquad (i = 1, 2, \cdots, n + 1). \tag{5}$$

The deformation D' will be defined as follows: For any value of the deforming parameter μ for which

$$0 < \mu \leqq 1$$

let each arc g_i on each curve of Z_m be divided into two successive segments whose arc lengths are in the ratio of μ to $1-\mu$. Corresponding to the given value of μ let the second of these two segments of g_i *be replaced* by itself but let the first of these two segments, say k_i, be replaced by that extremal segment, say h_i, that belongs to the class of extremals described in (C) and that joins k_i's end points. Points of h_i and k_i which divide h_i and k_i respectively in the same ratio as measured by arc lengths, shall be made to *correspond*, and shall accordingly be assigned the same values of t, namely the values already assigned to the points of k_i in the representation of Z_m in terms of t and P.

That we have here actually defined a deformation of Z_m readily follows from (C). Let Z_m' denote the final m-family of the deformation D'. According to paragraph (D) the curves of Z_m' will all lie in R_c, and according to paragraph (C), (c), their v coördinates will be functions of their u coördinates of class C at least. Further, the curves used in the deformation D' give to J at most the value which the corresponding curves of Z_m give, so that the curves used in D' satisfy (2).

We can now define a second deformation D'' under which Z_m' will be deformed into a family of canonical curves satisfying (2). The definition of D'' is similar to that of D' except that the segments g_i into which we divide the curves of Z_m' can here be defined in terms of u instead of s as follows:

$$u_{i-1} \leqq u \leqq u_i \qquad (i = 1, 2, \cdots, n + 1).$$

The combination of the deformations D' and D'' will obviously give a deformation of the desired sort and the lemma is proved except for its

concluding sentence. This concluding sentence is also readily seen to be true if the constant d of (D) be further restricted in magnitude so that the slopes of the canonical curves in Z_m will be less in absolute value than a sufficiently small positive constant.

In the deformation D'' a curve which initially was canonical is for each value of the deforming parameter μ replaced by itself. If in the original m-family the variable t had been u, then of the deformations D' and D'' only D'' would have been needed. We thus have the following lemma.

LEMMA 2. *If in Lemma* 1 *an m-family* Z_m *is given in which the variable* $t=u$, *then the deformation whose existence is affirmed in Lemma* 1 *can be defined in such a fashion that any curve of* Z_m *which is a canonical curve remains unaltered throughout the deformation.*

The following Lemma is important.

LEMMA 3. *Let* R_1 *be a region of the* (u, v) *plane enclosing* γ *in its interior, and in which the problem is regular. If* e' *be any sufficiently small positive constant, the region* $R_{e'}$ *consisting of points* (u, v) *within a distance* e' *of* γ *has the following property. If a canonical m-family* Z_m' *can be deformed within* $R_{e'}$ *into a canonical m-family* Z_m'' *through a deformation D all of whose curves give to J a value such that*

$$J \leqq J_0 - e^2, \tag{6}$$

where e is a positive constant, and J_0 *the value of J along* γ, *then* Z_m' *can be deformed into* Z_m'', *within* R_1, *through the mediation of curves all of which are canonical and which satisfy* (6).

For a moment suppose the region R_1 given in Lemma 1 is the region R_1 given in the present lemma. Corresponding to this choice of R_1 let $R_{a'}$ be a particular choice of the region R_a such that Lemmas 1 and 2 hold true. Now let us apply Lemma 1 again, this time taking for the region R_1 of Lemma 1 the region $R_{a'}$ just determined. Corresponding to this second choice of R_1, Lemma 1 will hold true for a second proper choice of R_a which we now denote by $R_{a''}$. We will prove Lemma 3 is true if $R_{e'}$ in Lemma 3 be taken as $R_{a''}$.

We first note that the complete set of curves used in the deformation D can be considered as an $(m+1)$-family Z_{m+1}. For if M_m is the common parametric manifold of Z_m' and Z_m'', each curve of D is specified by a pair (P, μ), where P is on M_m and μ on its interval $(0, 1)$. These curves of D can then be regarded as specified by a point $P_1=(P, \mu)$ the totality of which points form an $(m+1)$-dimensional manifold M_{m+1}.

Now the $(m+1)$-family Z_{m+1} can be deformed subject to (6), and within $R_{a'}$, into a canonical $(m+1)$-family $\bar{Z}_{m+1}$ using thereby the deformation, say $\bar{D}$, of Lemma 1. Under $\bar{D}$ the given canonical m-families Z_m' and Z_m'' appear initially in Z_{m+1}, and will be replaced finally in $\bar{Z}_{m+1}$ by canonical families, say $\bar{Z}_m'$ and $\bar{Z}_m''$ respectively. We will prove the lemma by showing how each of the first three canonical m-families of the set

$$Z_m', \bar{Z}_m', \bar{Z}_m'', Z_m'' \tag{7}$$

can be deformed, subject to (6), into its successor in (7) by a deformation in which only canonical curves are used.

(a) *The deformation of Z_m' into $\bar{Z}_m'$.* We may fix our attention upon those curves of $\bar{D}$ which serve to deform Z_m' into $\bar{Z}_m'$ and let Z_{m+1}' be the $(m+1)$-family of curves in $\bar{D}$ replacing Z_m', including also Z_m' and $\bar{Z}_m'$.

According to the concluding sentence of Lemma 1, Z_{m+1}' consists of curves along which v is a function of u of class C. According to Lemma 2, Z_{m+1} can then be carried by a deformation D' which does not alter Z_m' and $\bar{Z}_m'$ into a set of canonical curves. The final canonical $(m+1)$-family of curves of D' may be considered as the curves of the desired deformation, say D_1, of Z_m' into $\bar{Z}_m'$.

(b) *The deformation of $\bar{Z}_m'$ into $\bar{Z}_m''$.* The deformation, say D_2, required here is furnished by the set of canonical curves which make up $\bar{Z}_{m+1}$.

(c) *The deformation of $\bar{Z}_m''$ into Z_m''.* This deformation, say D_3, is set up as D_1 was set up in (a).

The combination of D_1, D_2, and D_3, in the order written, will give the required deformation of Z_m' into Z_m'' and the lemma is proved.

30. **The analysis situs of the deformation problem.** We have been denoting by $J(v_1, \cdots, v_n)$ the value of the integral J along the canonical curves joining the points (1) of §29. If $(u_0, 0)$ is not conjugate to $(u_{n+1}, 0)$ but if there are k conjugate points of $(u_0, 0)$ preceding $(u_{n+1}, 0)$, it follows from Theorem 2 of §6 that $J(v_1, \cdots, v_n)$ has a critical point of rank n and type k at the point $(v_1, \cdots, v_n) = (0, \cdots, 0)$. It follows from a lemma proved by the author* that there exists a one-to-one continuous transformation of the variables $(v_1, \cdots, v_n)$ into variables $(y_1, \cdots, y_n)$ in which the point $(v_1, \cdots, v_n) = (0, \cdots, 0)$ corresponds to $(y_1, \cdots, y_n) = (0, \cdots, 0)$, and under which

$$\begin{aligned} &J(v_1, \cdots, v_n) - J(0, \cdots, 0) \\ &\qquad = -y_1^2 - y_2^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_n^2 \end{aligned} \tag{1}$$

---

* Marston Morse, these Transactions, loc. cit., p. 354.

for $(y_1, \cdots, y_n)$ in the neighborhood of $(0, \cdots, 0)$. We exclude for the present the case where $k=0$.

For the sake of brevity we set

$$p^2 = y_1^2 + y_2^2 + \cdots + y_k^2, \qquad 0 < k \leqq n, \tag{2}$$

$$q^2 = y_{k+1}^2 + y_{k+2}^2 + \cdots + y_n^2, \tag{3}$$

understanding q^2 to be identically zero if $k=n$. Now let a be a positive constant so small that the domain

$$p^2 + q^2 \leqq a^2 \tag{4}$$

is one for which (1) holds as described.

In the present section we shall concern ourselves with deformations, in the neighborhood of the extremal segment γ, of m-families Z_m composed of canonical curves subject to a condition of the form

$$J(v_1, \cdots, v_n) \leqq J(0, \cdots, 0) - e^2, \quad e < a, \tag{5}$$

where e is an arbitrarily small positive constant. We shall confine ourselves to m-families corresponding to which the parametric manifold M_m is closed. Corresponding to each point P on M_m there is a curve on the given m-family Z_m. In the present case each such curve is canonical, and the points at which it crosses the straight lines $u=u_i$ have for their v coördinates v_i. Thus each point P on M_m determines a point $(v_1, \cdots, v_n)=(V)$. If we regard the points (V) as distinct when they arise from distinct points P on M_m, the set of all such points (V) form a closed m-dimensional manifold M_m' homeomorphic with the manifold M_m. If the given canonical m-family be deformed in the sense of §28, the manifold M_m' arising at each stage of the deformation from the m-family which replaces Z_m at that stage, will itself be "homotopically deformed" in the sense of analysis situs.*

We turn next to the space of the points $(Y)=(y_1, \cdots, y_n)$ and confine ourselves to a spherical neighborhood of the origin in the form (4), and to points in the space of the points (V) that correspond to the neighborhood in the space of the points (Y). Corresponding to a deformation of M_m' in the space of the points (V) we shall have to deal with a deformation of a corresponding manifold M_m'' in the space of the points (Y). Our requirement that the curves of the given m-family satisfy (5) becomes in terms of the points (Y) on the manifold M_m'' the condition

$$p^2 - q^2 \geqq e^2, \qquad 0 < k \leqq n. \tag{6}$$

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* Veblen, loc. cit., pp. 125–126.

31. **Homotopic deformations in the space (Y).** The preceding condition (6) and the restriction of the points (Y) to points in a spherical neighborhood of $(Y)=(0, \cdots, 0)$ combine to give us the relations

$$
(1) \qquad \begin{aligned} p^2 + q^2 &\leqq a^2, & 0 < e < a, \\ p^2 - q^2 &\geqq e^2, & 0 < k \leqq n. \end{aligned}
$$

The relations (1) define a finite closed region in the space of the points (Y). The deformation of m-families thus leads us to study deformations of closed manifolds in the space (1).

We could get some but not all of the properties of the space (1) by breaking it up into cells, and showing that of its connectivity numbers

$$R_0, R_1, \cdots, R_n,$$

all are unity except one, namely

$$R_{k-1} = 2.$$

Note that the $(k-1)$-dimensional sphere

$$
(2) \qquad p^2 = e^2, \qquad\qquad q^2 = 0
$$

is among the points of (1). Denote this sphere by S_{k-1}. A very important fact is that the region (1) can be deformed through the mediation of its own points into a singular complex on S_{k-1}. In terms of a deforming parameter μ the set of intermediate and final positions, $(y_1, \cdots, y_n)$, corresponding to any initial point $(a_1, \cdots, a_n)$ of (1), in a deformation D of (1) into a complex on S_{k-1} can be given as follows:

$$
(3) \qquad \begin{aligned} y_i^2 &= (1-\mu)a_i^2 + \frac{\mu e^2 a_i^2}{a_1^2 + \cdots + a_k^2} \qquad (i = 1, 2, \cdots, k), \\ y_i^2 &= (1-\mu)a_i^2 \qquad (i = k+1, k+2, \cdots, n), \end{aligned}
$$

where

$$
(4) \qquad 0 < \mu \leqq 1,
$$

and where the coördinates $(y_1, \cdots, y_n)$ are respectively required to have the sign of the coördinates $(a_1, \cdots, a_n)$ from which they are deformed.

We can now prove that any *m-dimensional closed manifold M_m, singular or non-singular, lying on* (1) *and such that m is less than $k-1$, can be deformed on* (1) *into a point.* For under the above deformation (3), M_m can be deformed into a manifold M_m' on S_{k-1}. Regardless of whether M_m' is singular or non-singular on S_{k-1}, there is a fundamental theorem on homotopic deformations* to the effect that M_m' can be deformed on S_{k-1} into a manifold

* Veblen, loc. cit., p. 131.

M_m'' on S_{k-1} each of whose cells "covers" a cell of S_{k-1}. Hence M_m'' can be deformed into a point on S_{k-1}; and the statement in italics follows at once.

The sphere S_{k-1} cannot itself be deformed on (1) *into a point on* (1). For if there were such a deformation, say D', let the family of intermediate manifolds homeomorphic with S_{k-1} under D' be deformed onto S_{k-1} under the deformation (3). The resulting family of manifolds on S_{k-1} would constitute a deformation on S_{k-1} in which the initial manifold would cover S_{k-1} just once and the final manifold coincide with a point. This, however, is impossible according to a fundamental theorem of analysis situs.*

We can now add the statement that S_{k-1} *cannot be deformed on* (1) *into any manifold on a manifold M_r on* (1) *of lower dimension than* $k-1$. For according to the preceding paragraph, M_r, and hence S_{k-1}, could be deformed on (1) into a point on (1). By making use of product manifolds analogous to surfaces of revolution in 3-space it is easy to set up for each $m>k-1$ an example of a manifold on (1) which cannot be deformed on (1) into any manifold of dimensionality less than $k-1$.

Finally with the aid of the deformation (3) we obtain the following result. *Any closed manifold M_m on* (1) *for which $m \geqq k-1$ can always be deformed on* (1) *into a manifold on S_{k-1}, and in special cases can be further deformed on S_{k-1} into a point.*

32. **The theorem on deformations of m-families joining A to B.** We can now translate the results of the preceding section into terms of deformations of canonical m-families. We shall understand that an *m-family shall be considered as "on" an r-family if every curve of the m-family coincides with some curve of the r-family.* Let it also be understood that a *closed m-family shall mean an m-family whose parametric manifold is closed.* In particular it should be noted that *a closed 0-family means a pair of ordinary curves joining the same two points.* The results of the preceding section now give the following lemma.

LEMMA. *Let γ be the extremal segment $v=0$ of* §29. *Suppose there are k points ($k>0$), conjugate to γ's initial point A and preceding γ's final point B, but that A is not conjugate to B. Then within any region R enclosing γ in its interior there exists a region R_1 enclosing γ in its interior, and corresponding to R_1 an arbitrarily small positive constant such that the closed m-families of canonical curves that satisfy*

$$J \leqq J_0 - e^2 \tag{1}$$

and lie in R_1 are conditioned as follows:

* Veblen, p. 131, §14.

(a) *Those for which* $m<k-1$ *can be deformed among canonical curves, within* R_1, *and subject to* (1), *into a single canonical curve.*

(b) *Those for which* $m\geqq k-1$ *include for each* m *at least one* m*-family that cannot be deformed among canonical curves, within* R_1, *and subject to* (1), *into a single canonical curve, or even into a* $(k-1)$*-family on an* r*-family for which* $r<k-1$.

(c) *Those for which* $m\geqq k-1$ *can always be deformed among canonical curves, within* R_1, *and subject to* (1), *into an* m*-family of canonical curves on a* $(k-1)$*-family* Z_{k-1}, *and, in special cases, can be further deformed into a single canonical curve.*

This lemma follows from the italicized statements of the preceding section, taking for Z_{k-1} the m-family determined by the points (V) that correspond to the points (Y) on S_{k-1}.

Lemmas 1, 2, 3 of §29 tell how and when any m-family of ordinary curves in the (u, v) plane can be deformed into a closed m-family of canonical curves. The preceding lemma describes the limitations on the deformations of closed m-families of canonical curves. The combination of the results of these lemmas carried over into the (x, y) plane gives the following fundamental theorem. (Here it will be convenient to call a region enclosing g in its interior, and consisting of points (x, y) within an arbitrarily small positive constant distance of g, *an arbitrarily small neighborhood of* g. The term *a sufficiently small neighborhood of* g will be similarly defined.)

THEOREM 9. *Let there be given in the* (x, y) *plane the extremal* g *of* §29.

On g *let there be* k *points conjugate to* A, $k>0$, *but suppose* B *is not conjugate to* A. *Let* J_0 *be the value of the integral* J *along* g.

Then corresponding to any sufficiently small neighborhood R *of* g, *there exists within* R *an arbitrarily small neighborhood* R_1 *of* g, *and an arbitrarily small positive constant* e *with the following properties. Closed* m*-families of curves which lie in* R_1, *which join* g*'s end points, and give to the integral* J *a value for which* (1) *holds, are conditioned as follows*:

(a) *Those for which* $m<k-1$ *can be deformed in* R *and subject to* (1) *into a single curve.*

(b) *Those for which* $m\geqq k-1$ *include for each* m *at least one* m*-family that cannot be deformed, within* R *and subject to* (1), *into any* $(k-1)$*-family on an* r*-family for which* $r<k-1$, *or into a single curve.*

(c) *Those for which* $m\geqq k-1$ *can always be deformed, within* R *and subject to* (1), *into an* m*-family on a* $(k-1)$*-family, and in special cases can be further deformed into a single curve.*

There remains the case where A is conjugate to B. Here we need to assume that the problem is analytic, as well as regular, in the neighborhood of g. We suppose that not every extremal through A with slope at A near that of g passes through B. In accordance with the conventions preceding Theorem 3, §12, g will have a type number $k=s$, or $k=s+1$, or type numbers $k=s$ and $s+1$, where s is the number of points on g conjugate to A and preceding B. *It can be shown that if $k=s$, or $k=s+1$, the preceding theorem holds if the sentence concerning conjugate points be omitted, while if g is of the composite type where $k=s$ and $k=s+1$, the preceding theorem should be further altered by replacing* (a), (b), *and* (c) *by the statement that any m-family whatsoever lying in R_1, joining A to B, and satisfying* (1) *can be deformed within R and subject to* (1), *into a single curve joining A to B.*

A proof of these last facts would lead us too far astray. The writer hopes to return to a discussion of critical points of rank less than n in a separate paper.

33. **An illustrative example.** Consider the equator of a unit sphere. Refer the sphere near C to coördinates (u, v), giving respectively the latitude and longitude of the point. Consider the sphere near C as now covered an infinite number of times by an unending strip S which overhangs the neighborhood of C after the manner of a Riemann surface. We may suppose S represented in the (u, v) plane by an unending strip R containing the u axis. We suppose our integral corresponds to the arc length on S.

The case $k=1$. If we take a segment of the u axis whose length g lies between π and 2π we will have the case $k=1$ of the preceding theorem. In the strip R we can clearly take two curves which join g's end points but otherwise lie on opposite sides of g, and whose lengths are equal but both less than g. These two curves form a 0-family which cannot be deformed into a single curve without passing out of R or increasing the lengths on S.

The case $k=2$. Suppose now that the length g of g is between 2π and 3π. Here we have the case $k=2$. Let u_0 and u_3 correspond to the end points of g, and let u_0, u_1, u_2, u_3 correspond to four successive points on the u axis such that no one of the resulting three successive intervals is as great as π in length. We consider a one-dimensional closed manifold of pairs (v_1, v_2), namely the pairs

$$v_1(a) = b\cos a, \quad v_2(a) = b\sin a,$$

where a is any real number, and is to be our parameter, and b is a positive constant. If b be sufficiently small, the following will hold true. On S the images of the points

$$[u_0, 0], \ [u_1, v_1(a)], \ [u_2, v_2(a)], \ [(u_3, 0]$$

can be successively joined by arcs of great circles which will give the shortest paths between their end points. The images on R of all such paths for all values of a will give a closed 1-family. The lengths of the different paths of the family considered as a function of a will have a maximum M which is readily seen to be less than g. If our choice of b be sufficiently small, it follows from the general theory that this 1-family cannot be deformed into a single curve joining g's end points without increasing the lengths beyond M or passing out of R.

If however we join the end points of g by any two ordinary curves whose lengths on S are less than a constant $g-e<g$, and which lie in a sufficiently small neighborhood of g, then, according to the preceding theorem, these two curves can be deformed into each other without passing out of R, or using curves whose lengths exceed $g-e$.

34. **Osgood's Theorem.** It has doubtless been noted that the case where $k=0$ has been omitted in the preceding discussion for the reason that there are in this case no curves neighboring the given extremal segment which satisfy the inequality (1) of the preceding section. For the case $k=0$ it seems to the author that the theorem that is due to Osgood* represents the nearest approach to the preceding developments. The questions at issue are sufficiently similar to have suggested to the author *a proof of Osgood's Theorem in the form stated by Hahn* (*B*, *p.* 281) *without, however, the hypothesis, used by Hahn and by recent writers on the calculus of variations, that at the ends of the extremal segment the problem be regular.* Such a proof seems desirable because it reduces the hypotheses under which Osgood's Theorem can be proved to a set identical with the least hypotheses under which the proper strong minimum is ordinarily established. This proof will be published later.

35. **Deformations of periodic extremals.** Although the results here are different from those of the preceding sections yet the methods may be carried over, and afford a proof of the desired results once the problem is well defined.

By an m-family of closed curves Z_m will be understood a set of ordinary closed curves in the (x, y) plane, the set of all of whose points make up a single-valued, continuous, point function of the point on a product complex C_{m+1}, obtained by combining an arbitrary point Q on a unit circle with an arbitrary point P on some m-dimensional manifold M_m. The dependence of the points of Z_m upon the points of C_{m+1} is to be such that each individual closed curve is obtained by holding P fast and varying Q on the unit circle;

* Osgood, these Transactions, vol. 2 (1901), p. 273.

and the dependence of the variable point of each individual curve upon Q, or more particularly upon the arc length measured from a fixed point on the unit circle to the point Q, is to be one by virtue of which the curve may be termed ordinary.

The remaining definitions and results of §§ 28, 29, 30, 31, and 32, may be carried over here either unchanged or with the changes obvious at each point. We have the following central theorem:

THEOREM 10. *Let there be given an integrand $F(x, y, \dot{x}, \dot{y})$ satisfying the hypotheses of* §1 *except that here F is to be of class C''''. Let there be given in S a non-degenerate* (§17) *periodic extremal g, at every point (x, y) of which, and for every θ, $F_1(x, y, \cos\theta, \sin\theta)$ is positive. Let k be the type number of g, determined as in Theorem* 5, §21. *Let J_0 be the value of J along g.*

Then corresponding to any sufficiently small neighborhood R of g, there exists within R an arbitrarily small neighborhood R_1 of g, and an arbitrarily small positive constant e with the following properties. Closed m-families of periodic curves which are deformable within R_1 into g, and which give to the integral J a value such that $J < J_0 - e^2$ satisfy the statements (a), (b), *and* (c) *of Theorem* 9.

36. **Birkhoff's Theorem.*** In dealing with deformations of one closed trajectory into another Birkhoff has given a theorem which for the case of dynamical systems is essentially equivalent to that special part of the preceding theorem which has to do with the mutual deformability of two closed trajectories. Birkhoff does not consider such entities as m-families in general. However his pair of closed trajectories comes under the head of what the author has called closed 0-families. Birkhoff's theorem is stated in terms of the Poincaré rotation number, and in terms of that rotation number tells when a pair of closed trajectories can be deformed into each other. *The author's theorem shows that periodic extremals for which $k > 1$ cannot be distinguished in type simply by a consideration of the mutual deformability of two closed trajectories.* It should be stated in explanation that Birkhoff was not seeking to distinguish between all types of periodic extremals by means of deformations, but rather to determine to what types his "minimax principle" applied, and for that purpose his theorem was sufficient.

Speaking generally, the results of Part VI show that *the number of conjugate points on extremals joining two fixed points, or the type number of periodic extremals, is the least integer m for which some $(m-1)$-family of curves*

* Birkhoff, loc. cit., p. 249.

cannot be deformed into a single curve subject to the conditions described in the text. This result suggests another new and powerful way of approaching a theory "in the large."

PART VII. VARIABLE END POINTS

37. **A geometric theorem.** No general development for this case will be presented. However, a suggestive and intrinsically complete geometric theorem has been obtained in a problem that comes under this head, namely the problem of the number and nature of the normals which can be drawn from a fixed point P in a euclidean $(n+1)$-space E_{n+1} to an n-spread S_n in that space. Proofs of the following results will be published later.

The n-spread S_n is to be a manifold without boundary in the sense of Veblen (loc. cit.). Moreover we assume that the points on S_n in the neighborhood of any given point of S_n are representable by an equation which gives one of the rectangular coördinates as a function of class C'' of the others. A preliminary result is that the loci of centers of principal normal curvature* C_n corresponding to S_n are not space filling in E_{n+1}, that is, there are in the neighborhood of each point of E_{n+1} points not on C_n. Let P be any point not on C_n or S_n. A second preliminary result is that any straight line L through P will be normal to S_n at no more than a finite number of points Q_i $(i=1, 2, \cdots, m)$. Each of the m segments PQ_i of L will be counted as a different normal from P to S_n. *A normal PQ_i will be said to be of type k if there are k centers of principal normal curvature corresponding to Q_i, on L, between P and Q_i.* We have the following theorem.

THEOREM. *Let P be any point not on the manifold S_n, nor on the loci of centers of principal normal curvature of S_n, and let M_k be the number of different normals from P to S_n of type k. Then between the numbers M_k, and the connectivities R_i of S_n, the following relations hold true:*

$$M_0 \geqq 1 + (R_0 - 1),$$

$$M_0 - M_1 \leqq 1 + (R_0 - 1) - (R_1 - 1),$$

$$M_0 - M_1 + M_2 \geqq 1 + (R_0 - 1) - (R_1 - 1) + (R_2 - 1),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$M_0 - M_1 + \cdots + (-1)^n M_n = 1 + (R_0 - 1) - + \cdots + (-1)^n (R_n - 1),$$

where the inequality signs alternate until the final equality is reached.

---

* Eisenhart, *Riemannian Geometry*, p. 153.

To this theorem* we add the result that points Q on S_n such that the straight line segment PQ gives a relative minimum or maximum to the distance from P to S_n, yield normals PQ, respectively of types 0 or n. We now make the following specific applications of the theorem.

In 3-space let S_2 be homeomorphic to a sphere. Here

$$R_0 = 1, \quad R_1 = 1, \quad R_2 = 2.$$

If there are r normals from P to S_2 which give a relative minimum to the distance from P to S_2, and s normals that give a relative maximum, *there are* $r+s-2$ *other normals of type* 1. The total number of normals is always even.

In 3-space let S_2 be homeomorphic to a torus. Here

$$R_0 = 1, \qquad R_1 = 3, \qquad R_2 = 2,$$
$$M_0 \geqq 1, \qquad M_1 \geqq 2, \qquad M_2 \geqq 1.$$

Thus there are always four normals at least. The total number of normals is always even.

In 4-space let S_3 be homeomorphic to a manifold obtained by identifying the opposite faces of an ordinary cube. Here

$$R_0 = 1, \qquad R_1 = 4, \qquad R_2 = 4, \qquad R_3 = 2,$$
$$M_0 \geqq 1, \qquad M_1 \geqq 3, \qquad M_2 \geqq 3, \qquad M_3 \geqq 1.$$

There are thus at least eight normals, while the total number of normals is always even.

* A theorem essentially the same as the above has been proved to hold in the general problem one variable end point in m dimensions.

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TOPOLOGICAL INVARIANTS OF KNOTS AND LINKS*

BY

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1. Introduction. The problem of finding sufficient invariants to determine completely the knot type of an arbitrary simple, closed curve in 3-space appears to be a very difficult one and is, at all events, not solved in this paper. However, we do succeed in deriving several new invariants by means of which it is possible, in many cases, to distinguish one type of knot from another. There exists one invariant, in particular, which is quite simple and effective. It takes the form of a polynomial $\Delta(x)$ with integer coefficients, where both the degree of the polynomial and the values of its coefficients are functions of the curve with which it is associated. Thus, for example, the invariant $\Delta(x)$ of an unknotted curve is 1, of a trefoil knot $1-x+x^2$, and so on. At the end of the paper, we have tabulated the various determinations of the invariant $\Delta(x)$ for the 84 knots of nine or less crossings listed as distinct in the tables of Tait and Kirkman. It turns out that with this one invariant we are able to distinguish between all the tabulated knots of eight or less crossings, of which there are 35. Repetitions of the same polynomial begin to appear when we come to knots of nine crossings.

The invariants found in this paper are all intimately related to the so-called *knot group*, as defined by Dehn. This is, of course, what one would expect; for many, if not all, of the topological properties of a knot are reflected in its group. The knot group would undoubtedly be an extremely powerful invariant if it could only be analyzed effectively; unfortunately, the problem of determining when two such groups are isomorphic appears to involve most of the difficulties of the knot problem itself.

In §11, we indicate, very briefly, how the results obtained for knots may be generalized to systems of knots, or links. We also establish the connection between the new invariants derived below and the invariants of the n-sheeted Riemann 3-spreads (generalized Riemann surfaces), associated with a knot.

2. Knots and their diagrams. In order to avoid certain troublesome complications of a point-theoretical order we shall always think of a *knot* as a simple, closed, sensed polygon in 3-space. A knot will, thus, be composed of a finite number of *vertices* and *sensed edges*. We shall allow ourselves to operate on a knot in the following three ways:

* Presented to the Society, May 7, 1927; received by the editors, October 13, 1927.

(i) To subdivide an edge into two sub-edges by creating a new vertex at a point of the edge.

(ii) To reverse the last operation: that is to say, to amalgamate a pair of consecutive collinear edges, along with their common vertex, into a single edge.

(iii) To change the shape of the knot by continuously displacing a vertex (along with the two edges meeting at the vertex) in such a manner that the knot never acquires a singularity during the process. It would, of course, be easy to express this third operation in purely combinatorial terms.

Two knots will be said to be the same *type* if, and only if, one of them is transformable into the other by a finite succession of operations of the three kinds just described. A knot will be said to be *unknotted* if, and only if, it is of the same type as a sensed triangle.

To make our descriptions a trifle more vivid we shall often allow ourselves considerable freedom of expression, with the tacit understanding that, at bottom, we are really looking at the problem from the combinatorial point of view. Thus, we shall sometimes talk of a knot as though it were a smooth elastic thread subject to actual physical deformations. There will, however, never be any real difficulty about translating any statement that we make into the less expressive language of pure, combinatorial analysis situs. In the figures, we shall picture a knot by a smooth curve rather than by a polygon. A purist may think of the curve as a polygon consisting of so many tiny sides that it gives an impression of smoothness to the eye.

A knot will be represented schematically by a 2-dimensional figure, or *diagram*. In the *plane of the diagram* a curve, called the *curve of the diagram*, will be traced picturing the knot as viewed from a point of space sufficiently removed so that the entire knot comes, at one time, within the field of vision. The curve of the diagram will ordinarily have singularities, but we shall assume that the point of observation is in a general position so that the singularities are all of the simplest possible sort: that is to say, double points with distinct tangents. The singularities of the curve of the diagram will be called *crossing points*, and the regions into which it subdivides the plane *regions of the diagram*. At each crossing point, two of the four corners will be dotted to indicate which of the two branches through the crossing point is to be

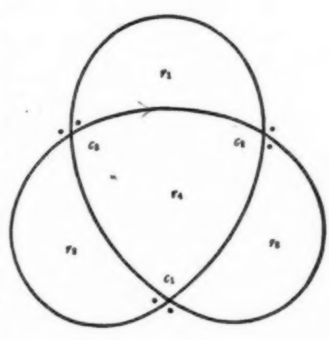

FIG. 1

thought of as the one passing under, or behind the other. The convention will be to place the dots in such a manner that an insect crawling in the positive sense along the "lower" branch through a crossing point would always have the two dotted corners on its left. Two corners will be said to be of *like signatures* if they are either both dotted or both undotted; they will be said to be of *unlike signatures* if one is dotted, the other not. Figure 1 represents a diagram of one of the two so-called trefoil knots.

To each region of a diagram a certain integer, called the *index* of the region, will be assigned. We shall allow ourselves to choose the index of any one region at random, but shall then fix the indices of all the remaining regions by imposing the requirement that whenever we cross the curve from right to left (with reference to our imaginary insect crawling along the curve in the positive sense) we must pass from a region of index p, let us say, to a region of next higher index $p+1$. Evidently, this condition determines the indices of all the remaining regions fully and without contradiction. To save words, we shall say that a corner of a region of index p is itself of index p.

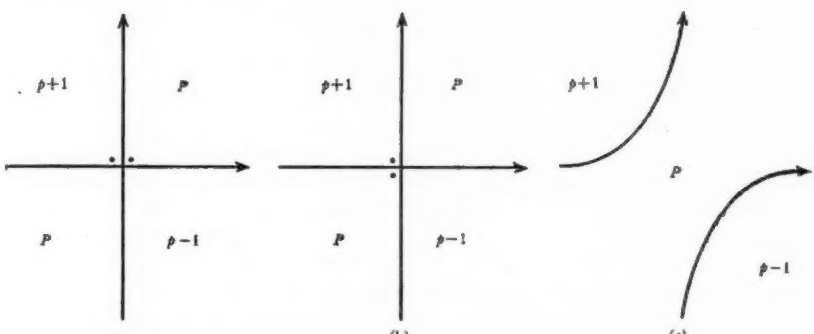

FIG. 2

It is easy to verify that at any crossing point c there are always two opposite corners of the same index p and two opposite corners of indices $p-1$ and $p+1$ respectively. The index p associated with the first pair of corners will be referred to as the *index* of the crossing point c. Two kinds of crossing points are to be distinguished according to which branch through the point passes under, or behind, the other. A crossing point of the first kind, Fig. 2a, will be said to be *right handed*, one of the second kind, Fig. 2b, *left handed*. At either kind of point the two undotted corners are of indices $p-1$ and p respectively, the two dotted ones of indices p and $p+1$. However, at a right

handed point the dotted corner of index p precedes the dotted corner of index $p+1$ as we circle around the point in the counter clockwise sense, whereas at a left handed point it follows the other. At a crossing point c, the two corners of like index p may belong to the same region of the diagram. We observe for future reference that on the boundary of a region of index p only crossing points of indices $p-1$, p, and $p+1$ may appear. Finally, we recall again that the entire system of indices is determined to within an additive constant only, since the index of some one region or crossing point has to be assigned before the indexing of the figure as a whole becomes determinate.

3. **The equations of a diagram.** In reality, the same diagram represents an infinite number of different knots, but this indetermination is, if anything, an advantage, as the knots so represented are all of the same type. The knot problem is the problem of recognizing when two different diagrams represent knots of the same type. Now, to tell the type of knot determined by a diagram it is evidently not necessary to know the exact shapes of the various elements of the diagram, but only the relations of incidence between the elements and the signatures at the corners of the regions. Because of this fact, the essential features of a diagram may all be displayed schematically by a properly chosen system of linear equations, as we shall now prove.

If a diagram has ν crossing points

$$c_i \qquad (i = 1, 2, \cdots, \nu), \tag{3.1}$$

we find, by a simple application of Euler's theorem on polyhedra, that it must have $\nu+2$ regions

$$r_j \qquad (j = 0, 1, \cdots, \nu + 1). \tag{3.2}$$

Now, suppose the four corners at a crossing point c_i belong respectively to the regions r_j, r_k, r_l, and r_m, that we pass through these regions in the cyclical order just named as we go around the point c_i in the counterclockwise sense, and that the two dotted corners are the ones belonging to the regions r_j and r_k respectively. Then, corresponding to the crossing point c_i we shall write the following linear equation:

$$c_i(r) = xr_j - xr_k + r_l - r_m = 0. \tag{3.3}$$

The ν equations (3.3) determined by the ν crossing points c_i will be called the *equations of the diagram*. The cyclical order of the terms in the left hand members of these equations plays an essential rôle and is not to be disturbed. The distribution of the coefficients x determines in which corners of the diagram the dots are located.

SUGGESTIONS TO AUTHORS

Much needless expense and many errors can be avoided. The editors of several mathematical journals have agreed upon the following suggestions.

1. Typewrite words and the very simplest formulas only.
2. *Do not* try to typewrite any complex formulas. Write them.
3. Keep a copy, and send the editors two copies, if you can.
4. *Do not* underline any symbols or any formulas.
5. Underline theorems with blue pencil (avoid ink).
6. Follow our recent styles in abbreviations, footnotes, etc.
7. Write carefully the (often misunderstood) capitals C K P S V W X Z.
8. Write ϵ, not ε. Write very carefully $\gamma\ \eta\ \kappa\ \lambda\ \nu\ \tau\ \upsilon\ \chi\ \omega$.
9. Among Greek capitals, use only $\Gamma\ \Delta\ \Theta\ \Lambda\ \Xi\ \Pi\ \Sigma\ \Phi\ \Psi\ \Omega$.
10. Punctuate carefully, especially in formulas; thus: $1, 2, \cdots, n$.
11. Use the solidus (/) to avoid fractions in solid lines.
12. Use fractional exponents to avoid root signs everywhere.
13. Use extra symbols to avoid complicated exponents.
14. In typewritten formulas, 1 means "one"; to indicate "ell" in formulas, backspace and overprint /; thus: ł. Similarly, 0 means "zero"; to indicate "cap O," backspace and overprint period; thus: ʘ.
15. Avoid a dash over a letter, except for those shown below.
16. Some samples of unusual types available on monotype machines follow. A more complete list of all such types will be sent on request.

Light Face Greek—$\alpha\ \beta\ \gamma \cdots$ (all) $\mathrm{A\ B\ \Gamma} \cdots$ (all).

★ Light Greek Superiors—$^{\Delta}$ and $^{\alpha\beta\gamma\cdots}$ (all except ι and o).

★ Light Greek Inferiors—$_{\Delta\Lambda\Sigma\Omega}$ and $_{\alpha\beta\gamma\cdots}$ (all except ι and o).

* Boldface Greek—$\boldsymbol{\alpha\ \beta\ \delta\ \epsilon\ \zeta\ \eta\ \theta\ \mu\ \nu\ \xi\ \pi\ \rho\ \sigma\ \omega}$ and $\boldsymbol{\Omega}$.

* Lightface German—$\mathfrak{a\ b\ c\ d\ p\ q\ A\ B\ C\ D\ E\ F\ G\ H\ I\ L\ M\ P\ O\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$.

* Boldface German—$\mathfrak{b\ A\ B\ D}$

Script (special font $\mathcal{A\ B\ C} \cdots$ (all). No lower case manufactured.

* Hebrew—ℵ א ב ב troublesome to handle.

★ Dashed Italics—$\bar{A}\ \bar{a}\ \bar{B}\ \bar{b}\ \bar{C}\ \bar{c}\ \bar{E}\ \bar{e}\ \bar{F}\ \bar{f}\ \bar{G}\ \bar{g}\ \bar{H}\ \bar{I}\ \bar{\imath}\ \bar{K}\ \bar{k}\ \bar{M}\ \bar{m}\ \bar{N}\ \bar{n}\ \bar{P}\ \bar{p}\ \bar{q}\ \bar{R}\ \bar{r}\ \bar{s}\ \bar{t}\ \bar{u}\ \bar{v}\ \bar{X}\ \bar{x}\ \bar{Y}\ \bar{y}\ \bar{Z}\ \bar{z}$

★ Tilda Italics—$\tilde{A}\ \tilde{a}\ \tilde{e}\ \tilde{N}\ \tilde{n}\ \tilde{O}\ \tilde{r}\ \tilde{u}\ \tilde{y}$

★ Tilda Greek—$\tilde{\alpha}\ \tilde{\epsilon}\ \tilde{\eta}\ \tilde{\omega}\ \tilde{\varpi}$

★ Dashed Greek—$\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}\ \bar{\delta}\ \bar{\eta}\ \bar{\theta}\ \bar{\mu}\ \bar{\nu}\ \bar{\rho}\ \bar{\omega}\ \bar{\Gamma}$

★ Dotted Italic—$\dot{a}\ \ddot{a}\ \dot{e}\ \ddot{e}\ \dot{g}\ \dot{\imath}\ \dot{m}\ \dot{n}\ \dot{q}\ \dot{r}\ \ddot{r}\ \dot{u}\ \ddot{u}\ \dot{x}\ \ddot{x}\ \dot{y}\ \ddot{y}\ \dot{z}\ \ddot{z}$.

★ Dotted Greek—$\dot{\eta}\ \ddot{\eta}\ \dot{\theta}\ \ddot{\theta}\ \dot{\xi}\ \dot{\psi}\ \ddot{\psi}\ \ddot{\omega}$ (single dotted $\zeta\ \phi\ \delta\ \beta\ \gamma$; double dotted γ readily available).

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* Additional characters readily available at small cost.

★ Matrices for additional characters are made upon special orders and necessitate a delay of from four to eight weeks and average expense of $4.50 per matrix.

SUGGESTIONS TO AUTHORS

Much needless expense and many errors can be avoided. The editors of several mathematical journals have agreed upon the following suggestions.

[illegible]

By way of illustration we shall write out the equations of the diagram of the trefoil knot (Fig. 1). They are as follows:

$$
\begin{aligned}
&c_1(r) = xr_2 - xr_0 + r_3 - r_4 = 0,\\
(3.4)\qquad &c_2(r) = xr_3 - xr_0 + r_1 - r_4 = 0,\\
&c_3(r) = xr_1 - xr_0 + r_2 - r_4 = 0.
\end{aligned}
$$

The equations of a diagram determine the structure of the diagram completely unless there happen to be two or more edges incident to the same pair of regions. For, barring this exceptional case, two cyclically consecutive terms in any equation correspond to a pair of regions that are incident along one edge only, and, therefore, determine the edge itself. In other words, the equations of the diagram tell us the incidence relations between the edges and crossing points. But they also tell us the relative position of the four edges at a crossing point; therefore, we have all the information needed to reconstruct the curve of the diagram. Moreover, the distribution of the coefficients x tells us how the corners must be dotted.

In the exceptional case, where the boundaries of two regions have more than one edge in common we are either dealing with the diagram of a *composite knot* K or with a diagram that admits of obvious simplification. Suppose the edges e_1 and e_2 are on the boundary of each of two regions r_1 and r_2. Then, if we join a point P_1 of the edge e_1 to a point P_2 of the edge e_2 by means of an arc α lying wholly within the region r_1, the extremities of the arc α will subdivide the curve of the diagram into two non-intersecting arcs γ_1 and γ_2 which may be combined respectively with the arc α to form the two closed curves

$$\alpha + \gamma_1, \qquad \alpha + \gamma_2.$$

Moreover, these last two curves may be regarded as the diagram curves of a pair of non-interlinking knots K_1 and K_2 in space. If neither of the knots K_1 nor K_2 is unknotted we may regard K_1 and K_2 as *factors* of the composite knot K. If one of them, K_1, is unknotted, the knot K must evidently be of the same type as the other one, K_2. Hence, in this case, the diagram of the knot K may be replaced by the simpler diagram of the knot K_2.

4. **The invariant polynomial $\Delta(x)$.** Let us now treat the equations of the diagram as a set of ordinary linear equations E in which the ordering of the terms in the various left hand members is immaterial. Then, the matrix of the coefficients of equations E will be a certain rectangular array M of ν rows and $\nu+2$ columns, one row corresponding to each crossing point and one column to each region of the diagram. We shall presently show that the

matrix M has a genuine invariantive significance; for the moment, let us merely observe that it has the following property:

If the matrix M is reduced to a square matrix M_0 by striking out two of its columns corresponding to regions with consecutive indices p and $p+1$, the determinant of the residual matrix M_0 will be independent of the two columns struck out, to within a factor of the form $\pm x^n$.

To prove the theorem, let us introduce the symbol R_p to denote the sum of all the columns corresponding to the regions of index p and the symbol 0 to denote a column made up exclusively of zero elements. Then, we obviously have the relation

$$\sum_p R_p = 0\,; \tag{4.1}$$

for in each row of the matrix there are only four non-vanishing elements, namely x, $-x$, 1, and -1, and the sum of these four elements is zero. We also have the relation

$$\sum_p x^{-p} R_p = 0\,; \tag{4.2}$$

for if we multiply the elements of each column by a factor x^{-p}, where p is the index of the (region corresponding to) the column, the four non-vanishing elements in a row of index q become x^{1-q}, $-x^{1-q}$, x^{-q} and $-x^{-q}$ respectively, so that their sum is again zero. By properly combining relations (4.1) and (4.2) we obtain the relation

$$\sum_p (x^{-p} - 1) R_p = 0 \tag{4.3}$$

in which the term in R_0 disappears.

Now, let

$$\pm\, \Delta_{pq}(x) = \pm\, \Delta_{qp}(x)$$

be the determinant of any one of the matrices M_{pq} obtained by striking out from the matrix M a pair of columns of indices p and q respectively. Then, by (4.3), we clearly have

$$(x^{-q} - 1)\Delta_{0p}(x) = \pm\, (x^{-p} - 1)\Delta_{0q}. \tag{4.4}$$

For relation (4.3) tells us that a column of index p multiplied by the factor $x^{-p}-1$ is expressible as a linear combination of the other columns of the matrix M of indices different from zero (that is to say, of columns of the matrix M_{0p}), and that in this linear combination the coefficients of the columns of index q are $-(x^{-q}-1)$. Moreover, since indices are determined to within an additive constant only, relation (4.4) gives us

$$(x^{r-q}-1)\Delta_{rp} = \pm\,(x^{r-p}-1)\Delta_{rq},$$

$$(x^{q-s}-1)\Delta_{qr} = \pm\,(x^{q-r}-1)\Delta_{qs}\,;$$

whence,

$$\Delta_{rp} = \pm\,\frac{x^{q-r}(x^{r-p}-1)}{x^{q-s}-1}\Delta_{qs}. \tag{4.5}$$

But, as a special case of (4.5), we have the relation

$$\Delta_{r(r+1)} = \pm\,x^{q-r}\Delta_{q(q+1)}, \tag{4.6}$$

which proves the theorem.

Let us now divide the determinant $\Delta_{r(r+1)}$ by a factor of the form $\pm x^n$ chosen in such a manner as to make the term of lowest degree in the resulting expression $\Delta(x)$ a positive constant. Then,

The polynomial $\Delta(x)$ is a knot invariant.

The theorem will be proved in §6 and again in §10, as a corollary to a more general theorem.

Let us actually evaluate the invariant $\Delta(x)$ in a simple, concrete case. From the equation of the diagram of the trefoil knot, (3.4), we obtain the matrix

$$\begin{matrix} -x & 0 & x & 1 & -1 \\ -x & 1 & 0 & x & -1 \\ -x & x & 1 & 0 & -1. \end{matrix} \tag{4.7}$$

Now, if we assign indices in such a way that the first row of the matrix is of index 2, the next three rows will be of index 1 and the last row of index 0. The determinant Δ_{01} obtained after striking out the last two rows of the matrix (4.7) will be

$$\Delta_{01} = -\,x(1-x+x^2)\,;$$

the determinant obtained after striking out the first two rows,

$$\Delta_{12}(x) = -\,(1-x+x^2).$$

The difference between these two expressions is of the sort predicted by relation (4.6). The invariant $\Delta(x)$ is, of course,

$$\Delta(x) = 1-x+x^2.$$

5. **Further new invariants.** It will now be necessary to obtain a somewhat more precise theorem about the matrix M than the one proved in §4.

Any two columns of the matrix M of consecutive indices p and $p+1$ may be expressed as linear combinations of the remaining ν columns, where the coefficients of the two linear combinations are polynomials in x with integer coefficients.

Here, and elsewhere throughout the discussion, we shall use the term "polynomial" in the broad sense, so as to allow terms in negative as well as positive powers of the mark x to be present.

Since indices are determined to within an additive constant only, we may assume that p is zero in proving the theorem. Now, in relation (4.3) there is no term in R_0, and the coefficient of the term in R_1 is $x^{-1}-1$. Let us divide the coefficients of all the terms in (4.3) by this last expression so as to make the coefficient of R_1 equal to unity. The coefficients of the remaining terms will then be expressible as polynomials in the broad sense; for if p is positive, we have

$$x^{-p} - 1/x^{-1} - 1 = x^{-p+1} + x^{-p+2} + \cdots + 1,$$

while if p is negative, we have

$$x^{-p} - 1/x^{-1} - 1 = - x^{-p} - x^{-p-1} - \cdots - x.$$

Therefore the simplified relation (4.3) tells us that any column of index 1 is expressible as a linear combination with polynomial coefficients of columns *of indices different from zero.* But if we start from the relation

$$\sum(x^{-p} - 1)R_p = 0$$

which also follows, at once, from (4.1) and (4.2), we conclude, by a similar argument, that any column of index 0 is expressible as a linear combination with polynomial coefficients of columns of *indices different from one.* The theorem follows at once.

Two matrices M_1 and M_2 will be said to be *equivalent* if it is possible to transform one of them into the other by means of the ordinary elementary operations allowed in the theory of matrices with integer coefficients:

(α) Multiplication of a row (column) by -1.

(β) Interchange of two rows (columns).

(γ) Addition of one row (column) to another.

(δ) Bordering the matrix with one new row and one new column, where the element common to the new row and column is 1 and the remaining elements of the new row and column are 0's; or the inverse operation of striking out a row and a column of the type just described.

Two matrices M_1 and M_2 will be said to be *ϵ-equivalent* if it is possible to transform one of them into the other by means of the operations (α), (β), (γ), (δ), along with the further operation

(ϵ) Multiplication or division of a row (column) by x. Two polynomials will be said to be ϵ-equivalent if they differ, at most, by a factor of the form $\pm x^n$. We now state the following theorem, which will be proved in the next section and again in §10.

If two diagrams represent knots of the same type their matrices M are ϵ-equivalent.

As a corollary to this theorem it follows that

If two diagrams represent knots of the same type the elementary factors of their matrices M are ϵ-equivalent, barring factors of the form $\pm x^p$ (those ϵ-equivalent to unity).

For operations (α), (β), and (γ) leave the elementary factors invariant; operation (δ) merely introduces or suppresses a unit factor; operation (ϵ) merely changes one of the factors by a factor x. By an *elementary factor* of a matrix M we here mean the highest common factor of all the minors of the matrix M of any given order p divided (for $p>1$) by the highest common factor of all minors of order $p-1$. If all the minors of order p vanish, the corresponding factor is zero.

The theorem about the invariance of the polynomial $\Delta(x)$ announced in §4 is an obvious consequence of the corollary, for $\Delta(x)$ is ϵ-equivalent to the product of the elementary factors of the matrix M.

The theorem suggests the problem of finding normal forms for the matrices M under operations (α), (β), (γ), (δ), and (ϵ). Under this particular group of operations, the elementary factors of a matrix M are not a complete set of invariants. They would be if we replaced operation (α) by the more general operation

(α') Multiplication of a row (column) by an arbitrary rational number.

The matrix M_0 obtained by deleting from the matrix M two columns of consecutive indices p and $p+1$ is ϵ-equivalent to the matrix M.

This follows, immediately, from the first theorem proved in this section.

It should be remarked that the matrix N obtained by changing the signs of all negative elements of the matrix M is equivalent to the matrix M. For if we change the signs of all the elements of M belonging to the columns of odd indices we obviously obtain a matrix of such a form that the elements in any given row are of like sign. Therefore, by further changing the signs of the elements in the rows containing no positive elements we obtain the matrix N. Thus, the matrix M is transformable into the matrix N by elementary transformations. For theoretical purposes, the matrix M offers certain advantages over the matrix N; when actual computations are to be

made, the matrix N is generally to be preferred, as mistakes in sign are less likely to be made when it is used.

6. **Diagram transformations.** When a knot is deformed, the equations of its diagram remain invariant so long as the topological structure of the diagram does not change. Now, a change in the structure of the diagram may come about in one or another of the following ways:

(A) The curve of the diagram may acquire a loop and crossing point (Fig. 3) or it may lose a loop and crossing point by a deformation of the inverse sort.

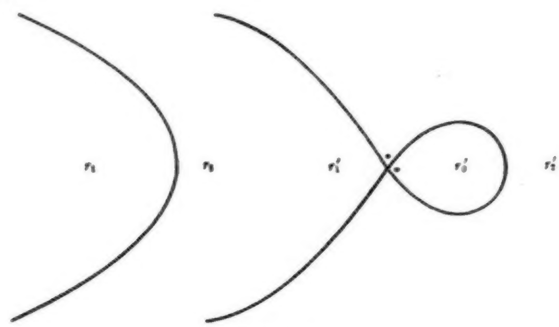

FIG. 3

(B) One branch of the curve may pass under another with the creation of two new crossing points (Fig. 4); or by a deformation of the inverse sort, one branch may slide out from under another with the loss of two cross-

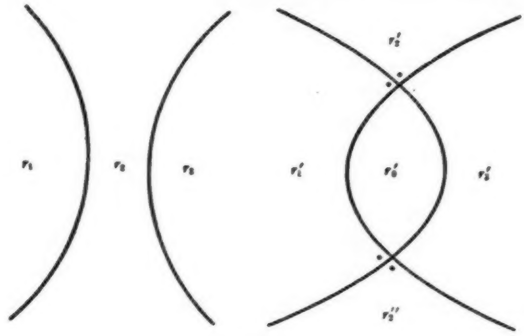

FIG. 4

ing points. In this case it must be borne in mind that the corners at the two crossing points must be so dotted as to imply that the lower branch at one is also the lower branch at the other.

(C) If there is a three-cornered region in the diagram, bounded by three arcs and three crossing points, and if the branch corresponding to one of the three arcs passes beneath the branches corresponding to the other two, then any one of the three branches may be deformed past the crossing point formed by the intersection of the other two (Fig. 5). The effect is the same, topologically speaking, whichever of the three branches undergoes the deformation.

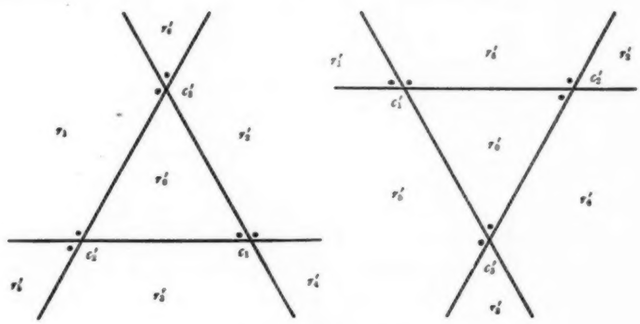

FIG. 5

It is a simple matter to verify that any allowable variation in the structure of the diagram may be compounded out of variations of the three simple types indicated above.*

With these facts before us, it is now easy to prove the theorem about the ϵ-equivalence of the matrices of two diagrams which determine knots of the same type. For it is sufficient to show that under each of the transformations (A), (B), and (C) the matrix of the diagram is carried into an ϵ-equivalent one.

First, consider case (A), where a branch of the curve acquires a new loop and crossing point. Let M_0 be the matrix of the original diagram after the two redundant columns corresponding to the region r_1 and r_2 (Fig. 3), have been struck out. Then, the effect of the transformation is merely to border the matrix M_0 with a new row and column in which all the elements are zero except the one which the row and column have in common. This last element will be ± 1 or $\pm x$ according to how the corners at the new crossing points are dotted. Evidently, the new matrix is ϵ-equivalent to the original one.

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* Cf., for example, Alexander and Briggs, *On types of knotted curves*, Annals of Mathematics, (2), vol. 28 (1927), pp. 563–586.

Under case (B), let M_0 be the matrix of the original diagram after the two redundant columns corresponding to the regions r_1 and r_2 (Fig. 4) have been struck out, and let M_0' be the corresponding transformed matrix. Then, if we add column r_0' of M_0' to column r_3' we obtain the original matrix M_0 bordered by two new rows and columns in the manner indicated schematically by the following figure:

$$\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & \Phi \\ \hline 0 & 0 & M_0 \end{array}.$$

Thus, clearly, the new matrix is again ϵ-equivalent to the old. It may happen that the corners are not dotted in the manner indicated in the figure, but that the two corners of the region r_0' are dotted ones. The method of proof is, however, essentially the same in this case as in the case just considered.

In disposing of case (C), we shall replace the matrix M of the original diagram by the equivalent matrix N, as defined at the end of §6, so as not to be troubled about the correct evaluation of the signs of the various elements. Moreover, we shall change the rôles of the rows and columns of the matrix N and think of this last as the matrix of a set of equations in the symbols c_i associated with the crossing points, rather than in the symbols r_j associated with the regions. Then, with the aid of Fig. 5, we may verify, at once, that the matrix N undergoes the transformation induced by the following change of symbols:

$$\begin{aligned} c_1' &= -xc_1, \\ c_2' &= xc_1 + c_3, \qquad (6.1) \\ c_3' &= c_1 + c_2. \end{aligned}$$

In making the verification we must not overlook the relations

$$xc_1 + c_2 + c_3 = 0$$

and

$$c_1' + xc_2' + xc_3' = 0$$

corresponding to the regions r_0 and r_0' (Fig. 5) respectively. Now, the substitution (6.1) is, clearly, the product of a substitution

$$c_1' = xc_1$$

which induces an ϵ-operation on the matrix N, and a substitution of determinant unity which induces a set of elementary operations, by a well known theorem. Therefore the matrix N is carried over into an ϵ-equivalent one. The cases where the corners are dotted in a different manner from that shown in Fig. 5 are treated in a similar manner.

This completes the proof of the invariantive character of the matrix M, whence, also, it follows that the polynomial $\Delta(x)$ is an invariant.

In the next sections we shall establish the connection between the matrix M and the so-called *group* of the knot, as defined by Dehn. This will necessitate the interpolation of a few preliminary remarks bearing, very largely, on questions of terminology and notation.

7. **Abstract groups.** In expressing the resultant of two or more operations of a group A we shall use the summation, rather than the product notation. Thus, if the symbols $a_i, a_j, \cdots$ represent operations of the group, the symbol $-a_i$ will represent the inverse of the operation a_i, the symbol a_i+a_j the resultant of the operation a_i followed by the operation a_j, the symbol 0 the identical operation. Furthermore, the symbol λa_i, where λ is any positive integer, will denote the resultant of the λ-fold repetitions of the operation a_i, and the symbol $-\lambda a_i$ the resultant of the λ-fold repetitions of the operation $-a_i$. It goes without saying that we must distinguish, in general, between the operations a_i+a_j and a_j+a_i.

If two consecutive terms of a sum of operations

$$c(a_i) = \lambda_1 a_{i_1} + \lambda_2 a_{i_2} + \cdots + \lambda_n a_{i_n}$$

involve the same letter a_{i_p} they may be contracted into a single term in a_{i_p}. After all possible contractions of the sort have been made the sum $c(a_i)$ will be said to be in its *reduced form*. We shall use the identity sign between two sums,

$$c(a_i) \equiv d(a_i),$$

to indicate that the sums, when reduced, are formally identical. An equality sign between two symbols

$$c = d$$

will merely indicate that the two symbols represent the same operation of the group, without implying their formal identity.

Let

$$a_i \qquad (i = 1, 2, \cdots, m) \tag{7.1}$$

be a set of *generators* of a group A: that is to say, a set of operations of the group A in terms of which all the operations of A may be expressed. Then, in most cases, there will exist certain identical combinations of the generators of the form

$$c_j(a_i) = 0 \qquad (j = 1, 2, \cdots, m). \tag{7.2}$$

Now, if we know any set of identities (7.2) among the generators there are three standard processes whereby we may enlarge the set (7.2) by the formation of new identities:

(i) The process of inversion, giving identities of the form

$$- c_j(a_i) = 0\,; \tag{7.3}$$

(ii) The process of summation, giving identities of the form

$$c_j(a_i) + c_k(a_i) = 0\,; \tag{7.4}$$

(iii) The process of transformation, giving identities of the form

$$e(a_i) + c_j(a_i) - e(a_i) = 0, \tag{7.5}$$

where $e(a_i)$ is any operation of the group.

The set (7.2) will be said to be *complete* if there is no identical combination of the generators which cannot ultimately be brought into the set by repeated application of the three processes just indicated. Thus, if the set (7.2) is complete, the most general identical combination of the generators must be of the form

$$c \equiv \sum_i (e_i \pm c_{j_i} - e_i) = 0. \tag{7.6}$$

A group A is fully determined by a set of generators (7.1) together with a complete set of identities (7.2) among them. In all of the discussion we shall confine our attention to the case when the number of generators (7.1) and defining identities (7.2) is finite.

With every group A there is associated a commutative group A_c determined by adjoining to the defining identities (7.2) of A all possible relations among the generators of the form

$$a_i + a_j = a_j + a_i.$$

The group A_c may also be thought of as the one determined by the generators (7.1) and identities (7.2) alone, where, however, we must now assign a new meaning to our symbolism and regard addition as commutative. If we do this, equations (7.2) simplify, by collecting terms in like symbols, to the form

$$c_j = \sum_i \epsilon_{ij} a_i = 0 \tag{7.7}$$

while relation (7.6) which displays the form of the most general identical combination among the generators becomes

$$c \equiv \sum c_j = 0. \tag{7.8}$$

The group A_c is, obviously, an invariant of the group A, whence, also, its own invariants are invariants of A. For future reference, we quote without proof a classical theorem about the commutative group A_c. Let $\|\epsilon_{ij}\|$ denote the matrix of the coefficients in equation (7.7). Then

The elementary factors of the matrix

$$\|\epsilon_{ij}\|$$

which differ from unity form a complete set of invariants of the group A_c. Therefore, also, they are invariants of the group A.

8. **The knot group.** The group R of a knot, as defined by Dehn,* is merely the ordinary topological group of the space S exterior to the knot. Let us fix upon some point of the space S, such as the observation point P from which we are supposed to be viewing the knot when we look at its diagram. Then, in the space S, each closed, sensed curve beginning and ending at P determines an operation of the group R. Moreover, two different sensed curves determine the same operation if, and only if, one may be deformed continuously into the other within the space S, while its ends remain fixed at the point P. During the deformation the curve may cut through itself at will, but it must never come into contact with the knot, as that would involve its leaving the space S. The condition that a curve determine the identical operation is that it be continuously deformable into the point P itself. If two sensed curves C_1 and C_2 correspond respectively to the operations r_1 and r_2 of the group R, the sensed curve C_1+C_2 obtained by joining the initial point of C_2 to the terminal point of C_1 determines the operation r_1+r_2.

The group R of a knot may be obtained, at once, from a diagram of the knot.† Let us flatten out the knot until it coincides, sensibly, with the curve of its diagram. Then, if we pick out, at random, a region r_0 of the diagram, there will be one generator of the group corresponding to each of the other $\nu+1$ regions r_j, where the generator in question is the operation determined by a curve which starts from the point P, crosses the region r_j, passes behind the plane of the diagram, and returns to the point P by way of the region r_0. It is easy to see, by inspection, that to each crossing point of the diagram there corresponds an identical relation of the form

$$r_j - r_k + r_l - r_m = 0, \tag{8.1}$$

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* M. Dehn, *Topologie des dreidimensionalen Raumes*, Mathematische Annalen, vol. 69 (1910), pp. 137–168.

† Dehn, loc. cit.

where if the symbol r_0 appears in this relation it must be set equal to zero. Moreover, it is not difficult to verify† that the set of relations (8.1) is complete. We, therefore, have the following theorem:

The group R of a knot is the one determined by the equations of the diagram, §3, *when we set x equal to unity, together with one more equation of the form*

$$r_0 = 0.$$

9. **Indexed groups.** We shall now make another short digression leading to a generalization of the theorem quoted at the end of §7. A group A will be said to be *indexed* if with each operation of the group there is associated an integer, called the *index* of the operation, such that

(i) The index of the identical operation is zero;

(ii) There exists an operation of index unity;

(iii) The index of the resultant of two operations is the sum of the indices of the two operations.

Two indexed groups will be said to be *directly equivalent* if they are related by a simple isomorphism pairing elements of like indices, and *inversely equivalent* if they are related by a simple isomorphism pairing elements of index p ($p=0, \pm 1, \pm 2, \cdots$) with elements of index $-p$.

Let A be an indexed group determined by a finite number of generators connected by a finite number of identical relations. Then, clearly, the generators may always be chosen in the *canonical form*

$$s, a_1, a_2, \cdots, a_n, \tag{9.1}$$

where the first generator s is of index 1 and the others a_i are of index 0. For any arbitrary finite set of generators may be reduced to the above form by a process analogous to the one used in finding the highest common factor of a set of integers. The defining identities of the group, expressed in terms of the generators (9.1), will be certain linear expressions which we shall denote by

$$c_j(s, a_1, a_2, \cdots, a_n) = 0. \tag{9.2}$$

Now, it will be observed that the operations of the group A which are of index 0 determine a self-conjugate subgroup A^* of A. Let a be any operation of this subgroup. Then if the operation a is expressed in terms of the generators (9.1) of the group A,

$$a = a(s, a_1, a_2, \cdots, a_n), \tag{9.3}$$

the sum of the coefficients of the terms in s must evidently vanish. Con-

† Dehn, loc. cit.

sequently, if we interpolate between every two terms of (9.3) a pair of redundant terms of the form $-\lambda s+\lambda s$, where the various coefficients λ are suitably determined, we shall obtain a representation of the operation a in the form

$$(9.4)\qquad a = \sum_i (\lambda_i s \pm a_{p_i} - \lambda_i s).$$

It will be convenient to introduce the *abridged notation*

$$\pm x^{\lambda} a_i = \lambda s \pm a_i - \lambda s$$

to denote a succession of three terms like the ones appearing in the sum (9.4). We shall then be able to express the sum a in the form

$$(9.5)\qquad a = \sum_i \pm x^{\lambda_i} a_{p_i}.$$

Conversely, every sum of terms $\pm x^{\lambda_i} a_{p_i}$ represents an operation of index 0: that is to say, an operation of the subgroup A^* of A. In particular, the defining relations (9.2) of the group A may be written

$$(9.6)\qquad c_j(x, a) = \sum_i \pm x^{\lambda_{ij}} a_{p_{ij}} = 0.$$

Now, let us reexamine the three processes (7.3), (7.4), and (7.5) of §7, whereby new identities are to be formed from the identities of a given set (9.6). The first two processes require no particular comment; in the new notation they may be expressed by

$$(9.7)\qquad - c_j(x, a) = 0$$

and

$$(9.8)\qquad c_j(x, a) + c_k(x, a) = 0$$

respectively. The third process, however, is expressed by

$$(9.9)\qquad e(x, a) + x^{\lambda} c_j(x, a) - e(x, a) = 0,$$

where the presence of the coefficient x^{λ} before the middle term is to be accounted for by the fact that in reducing expression (7.5) to the form (9.9) we must, in general, interpolate a pair of redundant terms $-\lambda s+\lambda s$ after the term $e(a_i)$ in order to obtain a group of terms

$$e(x, a) = e(a_i) - \lambda s$$

of index zero. Relation (7.6) which exhibits the most general identity among the generators (9.6) becomes, in the abridged notation,

$$(9.10)\qquad c(x, a) = \sum [e_i(x, a) \pm x^{\lambda_i} c_{j_i}(x, a) - e_i(x, a)] = 0.$$

We may now regard the subgroup A^* of A as determined by the n generators

(9.11) $$a_1, a_2, \cdots, a_n$$

of A of indices zero together with the identities (9.6), which, for convenience, we shall now rewrite:

(9.12) $$c_j(x, a) = 0.$$

Relation (9.10) shows us how to form the most general identity in the generators a_i.

With the group A^* there is associated a commutative group A_c^* determined by adding to the identities (9.12) all relations of the form

$$x^\lambda a_i \pm x^\mu a_j = \pm x^\mu a_j + x^\lambda a_i.$$

The group A_c^* bears the same relation to the group A^* as the group A_c of §7 to the group A. It may be thought of as the one determined by the generators (9.11) and identities (9.12) alone, where, as in §7, we change the meaning of our notation and regard addition as commutative. Here, however, we must bear in mind that it is only when we express the operations of the group A_c^* in abridged notation that addition is commutative. The operations a and

$$x^\lambda a = \lambda s + a - \lambda s$$

are still to be regarded as distinct, otherwise we would get only trivial results. The defining identities (9.12) of the commutative group A_c^* may evidently be simplified, by collecting the terms in the various symbols a_i, to the form

(9.13) $$c_j(x, a) = \sum_i X_{ij} a_i = 0,$$

where the coefficients X_{ij} are polynomials in x with integer coefficients. (Here, again, we are using the term "polynomial" in the broad sense so as to allow negative as well as positive powers of x to be present.) Relation (9.10) exhibiting the form of the most general identity in the generators simplifies, in this case, to

(9.14) $$c(x, a) = \sum_i X_i c_{j_i}(x, a) = 0,$$

where the coefficients X_i are also polynomials in the broad sense.

Now, let $\|X_{ij}\|$ be the matrix of the coefficients of equations (9.13). Then, as a generalization of the theorem quoted at the end of §7, we shall have the following proposition:

If two indexed groups A and B are directly equivalent, their associated matrices $\|X_{ij}\|$ *and* $\|Y_{ij}\|$ *are* ϵ*-equivalent.*

The proof will be made in a series of easy steps. Let us choose a canonical set of generators

$$s,\ a_1,\ a_2,\ \cdots,\ a_m \tag{9.15}$$

of the group A, and a set of defining identities

$$c_i(x,\ a) = 0 \qquad (xa = s + a - s). \tag{9.16}$$

Then, corresponding to these last, we shall have the identities

$$\sum X_{ij}a_i = 0 \tag{9.17}$$

of the commutative group A_ϵ^* associated with the group A.

Now, let us observe the following simple facts:

(i) If we enlarge the set (9.16) by the adjunction of one new relation which is dependent on the ones already in the set, the matrix $\|X_{ij}\|$ is, thereby, transformed into an ϵ-equivalent one. For, by relation (9.4), the matrix $\|X_{ij}\|$ merely acquires a new row, expressible as a linear combination of the old ones with polynomial coefficients.

(ii) If we adjoin a new generator a_{m+1} of index zero to the set (9.15) and, at the same time, add to relations (9.16) an identity of the form

$$c_{m+1}(x, a) + a_{m+1} = 0$$

expressing the new generator in terms of the old ones, the matrix $\|X_{ij}\|$ is again transformed into an ϵ-equivalent one. For the matrix $\|X_{ij}\|$ is merely bordered by a new row and column, where the elements of the new column are all zero except the one in the new row which is unity.

(iii) If we replace the generator s by another operation t of index unity such that the operations

$$t,\ a_1,\ a_2,\ \cdots,\ a_m \tag{9.18}$$

also generate the group A we leave the matrix $\|X_{ij}\|$ invariant. For suppose we write

$$y^\lambda a = \lambda t + a - \lambda t$$

to denote transformation through this new operation t. Then, since s and t are both of weight unity, it must be possible to write

$$s = t + \phi(y,\ a) \tag{9.19}$$

where $\phi(y,\ a)$ is of index zero and, therefore, expressible in the abridged notation. Therefore, we have

$$(9.20) \qquad xa = s + a - s = t + \phi + a - \phi - t = y(\phi + a - \phi).$$

But suppose we make the substitution (9.20) in equation (9.17). Since, in these last equations, addition is to be regarded as commutative, the substitution (9.20) produces the same effect as the substitution $x = y$. Therefore, the matrix $\|X_{ij}\|$ is left invariant, except for a change of notation.

With the above facts established, the proof of the theorem is immediate. Let the generators of the group B, written in canonical form, be

$$(9.21) \qquad t,\ b_1,\ b_2,\ \cdots,\ b_n,$$

and the defining relations,

$$(9.22) \qquad d_j(y, b) = 0.$$

Then, since the groups A and B are directly isomorphic, we may express the isomorphism either by the identities

$$(9.23) \qquad t = s + \phi(x, a) \qquad [yb = x(\phi + b - \phi)],$$

$$(9.24) \qquad b_i = \phi_i(x, a)$$

or by the inverted identities

$$(9.25) \qquad s = t + \psi(y, b) \qquad [xa,\ y(\psi + a - \psi)],$$

$$(9.26) \qquad a_i = \psi_i(y, b).$$

Now, starting with the generators (9.15) and identities (9.16), let us adjoin successively the generators b_i of the group B along with the corresponding relations (9.24) expressing these last in terms of the generators of the group A. Moreover, let us next adjoin successively relations (9.22) and (9.26), in which we think of y as expressed in terms of

$$x,\ a_1,\ \cdots,\ a_m,\ b_1,\ \cdots,\ b_n$$

by means of relation (9.23). We finally obtain the group A determined by the generators

$$s,\ a_1,\ \cdots,\ a_m,\ b_1,\ \cdots,\ b_n$$

with the defining identities (9.16), (9.22), (9.24) and (9.26). Moreover, by (ii) and (i), the matrix $\|X'_{ij}\|$ corresponding to this new mode of definition is ϵ-equivalent to the matrix $\|X_{ij}\|$.

By a similar argument, we may define the group B by means of the generators

$$t,\ a_1,\ \cdots,\ a_m,\ b_1,\ \cdots,\ b_n$$

along with the same defining identities (9.16), (9.24), and (9.26), where the matrix $\|Y'_{ij}\|$ corresponding to the new mode of definition is ϵ-equivalent

to the matrix $\|Y_{ij}\|$. Finally, by (iii) the matrix $\|X'_{ij}\|$ must be identical with the matrix $\|Y'_{ij}\|$ except for a change of notation. Therefore, the matrices $\|X_{ij}\|$ and $\|Y_{ij}\|$ must be ϵ-equivalent.

If two indexed groups A and B are inversely equivalent, the matrix $\|X_{ij}\|$ *associated with the group A goes over into a matrix which is ϵ-equivalent to the matrix* $\|Y_{ij}\|$ *associated with the group B if we make the change of marks* $x'=x^{-1}$.

The proof is similar to that of the previous theorem. The one essential difference is that in place of relation (9.19) we must write

$$s = -t + \phi(y, a) ;$$

whence, in place of (9.20), we have

$$xa = y^{-1}(\phi + a - \phi).$$

The matrices $\|X_{ij}\|$ *and* $\|X'_{ij}\|$ *corresponding to two different ways of defining an indexed group A are ϵ-equivalent. Moreover, the effect on the matrix of changing the signs of the indices of all the operations of an indexed group A is that produced by the substitution* $x'=x^{-1}$.

This is, of course, a consequence of the two previous theorems, when A and B are regarded as symbols for the same group.

10. **Application to knots.** The group R of a knot may evidently be thought of as an indexed group, for with each curve C determining an operation r of the group there is associated a certain integer measuring the number of times (in the algebraical sense) that the curve C winds around or loops the knot. This integer will be defined as the *index* of the operation r. With proper conventions as to what shall be the positive sense of winding around the knot, the index of an operation r_i of the group will evidently be equal to the index of the region r_i diminished by the index of the region r_0 or, if we choose the additive constant at our disposal so as to make the index of the last named region equal to zero, the index of the operation r_i will simply be the index of the region r_i.

Now, let us choose our notation so that r_0 and $r_{\nu+1}$ are two regions with consecutive indices 0 and 1. Moreover, let us denote by p_i the index of a general region r_i. Then, if we make the substitution

$$r_i = p_i s + r_i' \qquad (i = 1, 2, \cdots, \nu), \tag{10.1}$$

$$r_{\nu+1} = s$$

the new set of generators

$$s, r_1', r_2', \cdots, r_\nu'$$

will evidently be in canonical form, for the index of the first one will be unity and the indices of the others zero. Let us examine the form that the defining relations

$$r_i - r_j + r_k - r_l = 0 \tag{10.2}$$

of the group R take when written in the abridged notation. If an equation (10.2) corresponds to a right handed crossing point of index p it may be expressed as

$$ps + r_i' - r_j' - (p+1)s + ps + r_k' - r_l' - (p-1)s = 0$$

in terms of the canonical generators. Therefore, if we put

$$xr' = s + r' - s$$

it reduces, after we leave off the primes, to

$$xr_i - xr_j + r_k - r_l = 0.$$

A similar reduction leading to the same final result may be made if equations (10.2) correspond to a left-handed crossing point. Therefore,

The equations of the diagram taken in conjunction with two more equations of the form

$$r_0 = 0, \quad r_{\nu+1} = 0,$$

corresponding to regions with consecutive indices are the equations of the group of the knot written in abridged notation.

In other words, the matrix M_0, §5, obtained by striking out two columns of consecutive indices from the matrix M is simply the matrix $\|X_{ij}\|$, §9, of the group equations written in abridged notation. This gives us a second proof of the ϵ-invariantive character of the matrix M from which most of the other theorems in §§4 and 5 are immediately deducible.

11. **Links.** A *link* will be defined as a figure composed of the vertices and sensed edges of a finite number of non-intersecting knots. The most obvious link invariant is the number of knots into which the link may be resolved. We shall call this number the *multiplicity* μ of the link. A knot will thus be a link of multiplicity one. Evidently, the entire discussion up to this point applies not only to knots but to links of arbitrary multiplicities. That is to say, with every link there will be associated a matrix M having the same ϵ-invariantive significance as for the case of a knot, an invariant polynomial $\Delta(x)$, and so on. In the case of a link of higher multiplicity a broader generalization is, however, possible, as we shall now indicate.

Let L be a link of multiplicity μ made up of the elements of μ different knots

$$K_1, K_2, \cdots, K_\mu. \tag{11.1}$$

Then, at each crossing point c_i of the diagram of the link L the lower branch will belong to some knot K_a of system (11.1), the upper branch to some knot K_b which may, or may not be the same as the knot K_a. To the crossing point c_i we shall attach the number a associated with the knot K_a determined by the lower branch through the point. Moreover, we shall replace the equation (3.3) of the diagram associated with the crossing point c_i by a similar equation

$$x_a r_j - x_a r_k + r_l - r_m = 0, \tag{11.2}$$

where the coefficient x of the original equation has been replaced everywhere in the equation by the coefficient x_a. The matrix M_μ of the system of equations (11.2) determined by the various crossing points c_i will thus be an array in the marks 0, ± 1, and $\pm x_a$ $(a=1, 2, \cdots, \mu)$. It will reduce to the matrix M, as defined in §4, if we replace all of the marks x_a by one single mark x.

Two matrices M_μ will be said to be ϵ-equivalent if it is possible to transform one of them into the other by means of a finite number of elementary operations (α), (β), (γ), (δ), §5, in combination with a finite number of operations of the following type:

(ϵ) Multiplication or division of a row (column) by x_a.

This last operation is, of course, the natural generalization of the operation of the same name defined in §4 for the case where we have a simple mark x.

We now have the following broad generalization of one of the theorems of §5:

If two diagrams represent links of the same type their matrices M_μ are ϵ-equivalent.

The theorem may be verified directly by the elementary method of §6. It may also be derived by group theoretical considerations analogous to those developed in §§9 and 10. We shall indicate, briefly, the second method of proof.

The group A of a link of multiplicity μ is a *μ-tuply indexed group*; for with each operation a of the group there may be associated a composite index

$$(p_1, p_2, \cdots, p_\mu) \tag{11.3}$$

such that the number p_i is the linkage number of a curve determining the operation with the ith component knot K_i of the group. By a process

analogous to the one used in finding the highest common factor of a set of integers, it is easy to reduce any set of generators of the group A to the *canonical form*

$$(11.4) \qquad s_1, s_2, \cdots, s_\mu, a_1, \cdots, a_m,$$

where the index of each generator s_i of the first type is composed of zeros except for the number p_i which is one, and where the index of each generator a_i of the second type is composed exclusively of zeros. The identical relations in the generators will be certain linear expressions of the form

$$(11.5) \qquad c_j(s_1, \cdots, s_\mu, a_1, \cdots, a_m) = 0.$$

We shall denote by A' the group determined by the relations (11.5) together with all additional relations of the form

$$(11.6) \qquad s_i + s_j = s_j + s_i$$

expressing that any two generators (11.4) of the first type are commutative. Moreover, we shall denote by A^* the self conjugate subgroup of the group A' consisting of all operations of the group A' of index $(0, 0, \cdots, 0)$. Let us now use the *abridged notation*

$$(11.7) \qquad x_i a = s_i + a - s_i.$$

Then, by an obvious extension of the argument used in §9, we may show that the operations of the group A^* are precisely the ones which may be represented by sums of the form

$$\sum \pm x_1^{\lambda_{1ij}} x_2^{\lambda_{2ij}} \cdots x_\mu^{\lambda_{\mu ij}} a_{ij}.$$

Finally, if we impose a further set of relations making any two operations of the group A^* commutative, we obtain a group A_c^* consisting of all operations which may be represented in the form

$$(11.8) \qquad \sum_i X_i a_i,$$

where X_i is a polynomial in the marks $x_1, x_2, \cdots, x_\mu$. In the last expression, addition is, of course, to be regarded as commutative. The group A_c^* will be the one determined by the generators

$$a_1, a_2, \cdots, a_m$$

together with the identities

$$(11.9) \qquad \sum X_{ij} a_j = 0$$

to which the identities (11.5) reduce when expressed in the abridged notation and when addition is regarded as commutative.

By the methods of §10 it may be verified without difficulty that the matrix of the group A of a link is the matrix $\|X_{ij}\|$ of the coefficients in (11.8).

To obtain the direct generalization of the theory developed for knots, we must set

$$x_1 = x_2 = \cdots = x_\mu = x.$$

A certain number of easily calculable invariants are obtainable by setting all but one of the marks x_i equal to unity.

12. **Miscellaneous theorems.** Let the number of regions of a link diagram be $\nu+2$. Then, if the curve of the diagram is a connected point set the number of crossing points must be ν, just as the special case of a knot diagram. If the curve of the diagram is not a connected point set but is made up of $\kappa+1$ connected pieces, the number of crossing points is only $\nu-\kappa$. We must then adjoin to the equations of the diagram a set of κ equations of the form $0=0$ so that the matrices M and N, §4, shall have two less rows than columns. For we want the matrix M_0 from which we compute the invariant $\Delta(x)$ to be a square array of order ν. If the number κ is greater than unity the invariant $\Delta(x)$ evidently vanishes.

Several theorems are to be obtained by observing that when we set x equal to 1 in the matrix $M(x)=M$ the form of the resulting matrix $M(1)$ will be independent of how the corners at the crossing points are dotted and, therefore, independent of which branch through a crossing point is regarded as the one passing under the other. Suppose, for example, we start with the theorem that the invariant $\Delta(x)$ of an unknotted knot is unity, as may be verified, at once, by direct calculation. Then, as an immediate consequence, we may obtain to the following theorem about knots in general:

The sum of the coefficients of the invariant $\Delta(x)$ of a knot is always numerically equal to unity.

For, in the notation of §4, we have

$$\Delta(x) = \pm\, x^p \Delta_{r(r+1)}(x), \tag{12.1}$$

whence, the sum of the coefficients of the invariant $\Delta(x)$ must be given by

$$\Delta(1) = \pm\, \Delta_{r(r+1)}(1).$$

Now, by changing upper into lower branches at a suitably chosen set of crossing points we may always "unknot" the knot. For one obvious way of doing this is to reverse crossings in such a manner that if we start at a specified point P of the curve of the diagram and describe the curve in the positive sense we never pass through a crossing point along an upper branch without previously having passed through it along a lower one. But, as we

have already observed, such a reversal of crossings leaves invariant the value of the determinant $\Delta_{n(n+1)}(1)$. Therefore, since $\Delta(x)$ is unity for an unknotted knot, we have

$$1 = \pm \Delta_{r(r+1)}(1),$$

which proves the theorem.

The multiplicity μ of a link L is equal to the number of zero elementary factors of the matrix $M(1)$ obtained by setting x equal to 1 in the matrix M.

For, by an elementary calculation, we verify that the theorem is true when the link L consists of μ unknotted and non-interlinking curves.

It should be remembered that if we assign a constant integral value c to x in the matrix M and then derive the elementary factors of the matrix $M(c)$, regarding the latter as a matrix in integer elements, we do not necessarily get the same result as if we derived the elementary factors of the matrix $M(x)$ regarded as a matrix in polynomial elements and then substituted the value c for x in these factors. For in calculating the factors of the matrix $M(x)$ we have to allow the operation of adding to one row (column) a rational multiple of another row (column), which operation is not allowed in calculating the factors of the matrix $M(c)$ unless the rational multiple is also integral. It may, therefore, be worth while noting the following theorem:

The elementary factors of the matrix $M(x)$ (c integral) that are not of the form $\pm c^p$ are all link invariants to within a multiple of c.

For the matrix operations (α), (β), and (γ) of §4 leave the elementary factors of $M(c)$ unaltered, the operation (δ) merely adds or takes away a unit factor, and the operation (ϵ) merely multiplies or divides one factor by c.

Let $\nu+2$ be the number of regions of a link diagram and p the number of distinct values taken on by the indices of the various crossing points (corresponding to any one way of assigning indices). Then the degree of the polynomial $\Delta(x)$ never exceeds $\nu-p$.

For, in the notation of §4, the degree of the polynomial $\Delta(x)$ never exceeds that of the polynomial $\Delta_{p(p+1)}(x)$. Moreover, by (4.6) we have

$$\Delta_{01}(x) = \pm x^p \Delta_{p(p+1)}(x).$$

But the degree of $\Delta_{01}(x)$ cannot exceed ν since this expression is a ν-rowed determinant with elements that are linear in x. Therefore, the degree of $\Delta_{p(p+1)}(x)$, and consequently also of $\Delta(x)$, never exceeds $\nu-p$.

If K is a composite knot, §3, *made up of the factors K_1 and K_2, the invariant $\Delta(x)$ of the knot K is equal to the product of the corresponding invariants of the knots K_1 and K_2.*

A composite knot is one which may be deformed in such a way that its diagram will be of the sort considered in the last paragraph of §3, where two edges e_1 and e_2 appear on the boundaries of two different regions r_1 and r_2 of the diagram. In the notation of §3, let $\alpha+\gamma_1$ and $\alpha+\gamma_2$ be the curves of the diagram of the two factors K_1 and K_2 of K. Moreover, let $\Delta_{r(r+1)}(x)$ be the determinant of the knot K after elimination from its matrix M of the two redundant columns corresponding to the regions r_1 and r_2 respectively. Thus, clearly the determinant $\Delta_{r(r+1)}(x)$ is equal to the product of the two similar determinants $\Delta'_{r(r+1)}(x)$ and $\Delta''_{r(r+1)}(x)$ corresponding to the two factors K_1 and K_2; for the determinant $\Delta_{r(r+1)}(x)$ is related to these other two in the manner illustrated schematically by

$$\Delta_{r(r+1)}(x) = \left|\begin{array}{c|c} \Delta'_{r(r+1)} & 0 \\ \hline 0 & \Delta''_{r(r+1)} \end{array}\right|.$$

The theorem therefore follows, at once.

We shall bring these miscellaneous remarks to a close by obtaining a relation between the polynomial invariants $\Delta(x)$ of three closely associated links. Consider a link diagram Z' with a right handed crossing point of index p (Fig. 2a). By changing this one crossing point into a left handed one (Fig. 2b) we obtain a new diagram Z''. Moreover, by cutting the two branches at the crossing point, separating them slightly, and rejoining them so as to unite into one the two regions of index p incident to the crossing point (Fig. 2c) we obtain a third diagram Z. Now, suppose we form the matrix N (§4) of the diagram Z' and arrange the rows and columns in such an order that the first row corresponds to the crossing point c and that the first four columns correspond to the four regions incident to the point c and represented in Fig. 2a by the first, second, third and fourth quadrants respectively. The first row of the matrix N' will, thus, start with the elements x, x, 1, 1 followed by zeros. To obtain the corresponding matrix N'' of the diagram Z we shall merely have to interchange the first and third elements in the first row of the matrix N'; to obtain the corresponding matrix N of the diagram Z we shall merely have to add the first column of the matrix N' to the third and then strike out the first row and first column. Now, let $\Delta'_{p(p+1)}$, $\Delta''_{p(p+1)}$ and $\Delta_{p(p+1)}$ be the determinants of the matrices obtained by striking out the first two columns of N', N'', and N, respectively. Then, clearly, the determinant $\Delta_{p(p+1)}$ will be the principal minor of both the determinants $\Delta'_{p(p+1)}$ and $\Delta''_{p(p+1)}$. Let Γ be the minor of the second element

in the first row of $\Delta'_{p(p+1)}$ (and of $\Delta''_{p\,(p+1)}$). Then if we expand each of the determinants $\Delta'_{p(p+1)}$ and $\Delta''_{p\,(p+1)}$ in terms of the elements of its first row and their minors, we find

$$\Delta'_{p(p+1)} = \Delta_{p(p+1)} - \Gamma, \quad \Delta''_{p(p+1)} = x\Delta_{p(p+1)} - \Gamma,$$

whence,

$$\Delta'_{p(p+1)} - \Delta''_{p(p+1)} = (1 - x)\Delta_{p(p+1)}, \tag{12.2}$$

which is the relation we had in mind to establish.

The argument needs to be slightly modified if the two corners of index p at the crossing point c belong to the same region of the diagram. In this case, however, the curve of the diagram Z must be a disconnected point set, whence,

$$\Delta_{p(p+1)} = 0.$$

Moreover, we obviously have

$$\Delta'_{p(p+1)} = \Delta''_{p(p+1)},$$

so that relation (12.2) continues to hold.

13. **n-sheeted spreads.** Further invariants of the matrix M are to be obtained by regarding each element as the symbol for a certain square array of order n, where the connection between the symbols and the arrays which they represent is as follows:

(i) 0 is the symbol for an array composed entirely of zeros.

(ii) 1 is the symbol for an array with 1's along the main diagonal and 0's elsewhere.

(iii) x is the symbol for an array obtained from the array 1 either by permuting the columns cyclically so that the first column goes into the second or by permuting the rows cyclically so that the second row goes into the first; the effect of either permutation is the same. x^p is the symbol for the array obtained from the array 1 by making p successive permutations of the type just described.

Now, if we replace each element of the matrix M by the array which it symbolizes we obtain a new array M^n of νn rows and $(\nu+2)n$ columns, made up of integer elements.

The elementary factors of the array M^n that differ from unity are link invariants.

For to an elementary operation (α), (β), (γ), or (δ) on the matrix M there evidently corresponds an operation on the matrix M^n which may be resolved into elementary operations. Moreover, to an extended operation (ϵ) on

the matrix M there merely corresponds a cyclical permutation of a block of n rows (columns) of the matrix M^n, which may again be resolved into elementary operations. This proves the theorem.

Corresponding to a pair of redundant columns of the matrix M there will be two sets of redundant columns of the matrix M^n composed of n columns each. We may, therefore, always replace the matrix M^n by a square matrix M_0^n of order νn.

The elementary factors of the array M_0^n have an interesting geometrical significance. If the link from which we start is a knot, the number of zero factors is equal to the connectivity numbers,

$$(P_1 - 1) = (P_2 - 1),$$

of an n-sheeted Riemann spread S^n (the 3-dimensional generalization of an n-sheeted Riemann surface) with the knot as branch curve (generalized branch point). The divisors that are different from zero and unity are the *coefficients of torsion* of the spread S^n.* This may all be proved very easily by making a suitable cellular subdivision of the spread S^n, as we shall now indicate.†

Let K be the knot under consideration and S the space containing it, where to simplify matters, we shall suppose that the space S closes up to a single point at infinity. Then, the first step will be to cut up the space S into cells, in the following manner. Wherever one branch of the knot appears to pass behind another, we shall join the upper branch to the lower one by a segment c_i'. The ends of the segment c_i' will be the vertices, or 0-cells, of the subdivision; the segments c_i' themselves, together with the arcs a_i' into which the ends of the segments c_i' subdivide the knot will be the 1-cells. Corresponding to each region r_i of the diagram we shall construct a 2-cell r_i' of which r_i will be the projection, bounded by the appropriate arcs a_i' and c_i'. The residual part of the space S will then consist of a pair of 3-cells. Thus, to sum up, the subdivision Σ that we have just described will consist of 2ν vertices, 3ν edges, $\nu+2$ 2-cells, and 2 3-cells. Corresponding to the subdivision Σ of the space S there will be a subdivision Σ^n of the spread S^n

* In a paper read before the National Academy in November, 1920, cf. Veblen, Cambridge Colloquium Lectures (1922), *Analysis Situs*, p. 150, I made the observation that the topological invariants of the n-sheeted spreads associated with a knot would be invariants of the knot itself, and showed by actual calculation that these invariants could be used to distinguish between a number of the more elementary knots. Later, these same invariants were discovered independently, by F. Reidemeister, *Knoten und Gruppen*, Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität, 1926, pp. 7–23. See also a paper by Alexander and Briggs. *On types of knotted curves*, Annals of Mathematics, (2), vol. 28 (1927), pp. 563–586.

† Cf. Alexander and Briggs, loc. cit.

such that each cell of the subdivision Σ which is not composed of points of the knot will be covered by n cells of the subdivision Σ^n, one in each sheet of the spread S^n, and such that each cell of the subdivision Σ which is composed of points of the knot will be covered by just one cell of the subdivision Σ^n. Along the knot, of course, the n sheets of the spread S^n merge into one.

We may simplify the subdivision Σ^n somewhat by amalgamating all but one of the 2ν edges covering the edges a_i', along with their end points, into one single 0-cell; or, if we prefer, by treating this group of 0-cells and 1-cells as a single generalized vertex A. The remaining 1-cells of the subdivision will then represent closed 1-circuits beginning and ending at the vertex A, while the boundaries of the various 2-cells will give the relations of bounding among these circuits. Clearly the number of 1-circuits will be $\nu n+1$, as there will be n circuits

$$c_{ij}' \qquad (j = 1, 2, \cdots, n), \tag{13.1}$$

corresponding to each segment c_i', together with one additional circuit c corresponding to the residual arc of the branch system that has not been amalgamated into the generalized vertex A. Moreover, there will be $\nu n+2$ relations among these circuits corresponding to the $\nu n+2$ 2-cells

$$r_{ij}' \qquad (j = 1, 2, \cdots, n) \tag{13.2}$$

covering the 2-cells r_i' of the subdivision Σ.

Now, it is easy to verify that the matrix M^n is precisely the one exhibiting the incidence relations between the 1-circuits (13.1) and 2-cells (13.2). Therefore, to obtain a matrix exhibiting the incidence relations between all the 1-circuits and 2-cells we have only to adjoin to the matrix M^n a new row corresponding to the remaining 1-circuit c. As the 1-circuit c_i is on the boundary of two blocks of n 2-cells each, corresponding to a pair of contiguous regions r_0 and $r_{\nu+1}$ of the diagram, the added row will consist of a block of n 1's, a block of n -1's, and zeros. The elementary factors of the matrix with the added row will be the ones that determine the connectivity numbers and coefficients of torsion of the spread S^n. But this last matrix is evidently equivalent to the matrix M_0^n, for the $2n$ columns having the elements ± 1 in the last row correspond to the redundant columns of the matrix M^n; hence, by adding to these columns suitable linear combinations of the remaining νn ones we may make all their elements zero except the ones in the last row. Thus, the geometrical interpretation of the divisors of the matrix M_0^n is established.

Since the matrix with the added row may be transformed into one such that in certain columns all the elements will be zeros with the exception of

an element 1 in the last column, it follows that the 1-circuit c must be a bounding curve,

$$c \sim 0\,;$$

for each column determines a relation of bounding among the 1-circuits determined by the rows. But the 1-circuit c, when we include the generalized point A which really forms a part of it, is simply the branch curve of the spread S^n itself. Hence we have the theorem†

The branch curve of the n-sheeted spread determined by a knot K is always a bounding curve of the spread.

The geometrical interpretation of the factors of the matrix M^n is not quite so satisfactory for the case of a link of multiplicity greater than unity. However, by a similar argument to the one made above, it is easy to show that these factors give the connectivity numbers and coefficients of torsion of the spread S^n when we treat the entire branch system of S^n as if it were a single generalized point.

14. **Tabulation of $\Delta(x)$.** At the end of the paper referred to in the last footnote a chart has been drawn up showing diagrams of the eighty-four knots of nine or less crossings listed as distinct by Tait and Kirkman; also a table giving the torsion numbers of the 2- and 3-sheeted Riemann spreads

3_{1a}	1−1	7_{7a}	1−5+9	9_{7a}	3−7+9	9_{11a}	1−5+7−7
4_{1a}	1−3	9_{45n}	1−6+9	8_{15a}	3−8+11	9_{17a}	1−5+9−9
5_{2a}	2−3	9_{47n}	1−7+11	9_{25a}	3−12+17	9_{20a}	1−5+9−11
6_{1a}	2−5*	8_{12a}	1−7+13	9_{41a}	3−12+19	9_{22a}	1−5+10−11
9_{46n}		7_{3a}	2−3+3	9_{39a}	3−14+21	8_{18a}	1−5+10−13*
7_{2a}	3−5	7_{5a}	2−4+5	9_{10a}	4−8+9	9_{24a}	
8_{1a}	3−7	8_{4a}	2−5+5	9_{13a}	4−9+11	9_{26a}	1−5+11−13
7_{4a}	4−7⁺	8_{6a}	2−6+7	9_{18a}	4−10+13	9_{27a}	1−5+11−15
9_{2a}		8_{8a}	2−6+9	9_{23a}	4−11+15	9_{28a}	1−5+12−15⁺
8_{3a}	4−9	8_{11a}	2−7+9	9_{38a}	5−14+19	9_{29a}	
9_{5a}	6−11	8_{13a}	2−7+11	8_{19n}	1−1+0+1	9_{30a}	1−5+12−17
9_{35a}	7−13	8_{14a}	2−8+11⁺	7_{1a}	1−1+1−1	9_{31a}	1−5+13−17
5_{1a}	1−1+1	9_{8a}		9_{43n}	1−3+2−1	9_{32a}	1−6+14−17
9_{42n}	1−2+1	9_{12a}	2−9+13	8_{2a}	1−3+3−3	9_{33a}	1−6+14−19
8_{20n}	1−2+3	9_{14a}	2−9+15	8_{5a}	1−3+4−5	9_{34a}	1−6+16−23
6_{2a}	1−3+3	9_{15a}	2−10+15	8_{7a}	1−3+5−5	9_{40a}	1−7+18−23
6_{3a}	1−3+5	9_{19a}	2−10+17	8_{9a}	1−3+5−7	9_{3a}	2−3+3−3
8_{21n}	1−4+5*	9_{21a}	2−11+17	8_{10a}	1−3+6−7	9_{6a}	2−4+5−5
9_{36a}		9_{37a}	2−11+19	9_{48n}	1−4+6−5	9_{9a}	2−4+6−7
9_{44n}	1−4+7	9_{4a}	3−5+5	8_{16a}	1−4+8−9	9_{16a}	2−5+8−9
7_{6a}	1−5+7	9_{49n}	3−6+7	8_{17a}	1−4+8−11	9_{1a}	1−1+1−1+1

† A and B, loc cit.

associated with the knots. We list below the values of the polynomial $\Delta(x)$ for these same knots. It appears that the polynomial $\Delta(x)$ of a knot is always of even degree and that its coefficients are arranged symmetrically with reference to the middle one. Therefore, to reduce the space occupied by the table we have merely indicated the values of the coefficients of $\Delta(x)$ up to, and including, the middle one. To illustrate how the table is to be used, let us turn, for example, to the entry

$$\|9_{33a} \mid 1 - 6 + 14 - 19\|.$$

This indicates that an alternating knot of nine crossings listed as the thirty-third in the chart referred to above has the polynomial invariant

$$\Delta(x) = 1 - 6x + 14x^2 - 19x^3 + 14x^4 - 6x^5 + x^6.$$

In the table, six repetitions of the same polynomial appear. In the three cases marked with (*) the two paired knots may be distinguished with the aid of other invariants of the matrix M, such, for example, as the coefficients of torsion of the associated Riemann spaces. In the three cases marked with (+) the two paired knots have ϵ-equivalent matrices M and, therefore, cannot be distinguished (assuming that they actually are distinct, which Tait never really proves) by the methods of this paper.

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ON THE EXPANSION OF ANALYTIC FUNCTIONS IN SERIES OF POLYNOMIALS AND IN SERIES OF OTHER ANALYTIC FUNCTIONS*

BY

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1. Introduction. The present paper is substantially a continuation of a previous paper† in which polynomial developments of an arbitrary analytic function were considered, culminating in three theorems. The first of these theorems is in essence a modification and completion of a result due to Birkhoff, a generalization of Taylor's development about the origin in the plane of the complex variable x:‡

THEOREM I. *Let the functions*

$$p_0(x),\ p_1(x),\ p_2(x),\ \cdots$$

be analytic for $|x|\leqq 1+\epsilon$, *and such that on and within the circle* γ': $|x|=1+\epsilon$, *we have*

$$(1)\qquad |\,p_k(x)-x^k\,|\leqq \epsilon_k \qquad (k=0,1,2,\cdots),$$

where the series $\sum\epsilon_k^2$ *converges to a sum less than unity, and where the series* $\sum\epsilon_k$ *converges. Then there exists a set of functions* $P_k(x)$ *continuous for* $|x|\geqq 1$, *analytic for* $|x|>1$,§ *zero at infinity, and such that*

$$(2)\qquad \int_\gamma P_k(x)p_i(x)dx=\delta_{ki}=\begin{cases}0, & i\neq k,\\ 1, & i=k,\end{cases}\qquad \gamma:\ |\,x\,|=1.$$

If $F(x)$ is any function integrable and with an integrable square (in the sense of Lebesgue), then the two series

$$(3)\qquad \sum_{k=0}^{\infty}a_kx^k,\qquad a_k=\frac{1}{2\pi i}\int_\gamma F(x)x^{-k-1}dx,$$

$$(4)\qquad \sum_{k=0}^{\infty}c_kp_k(x),\qquad c_k=\int_\gamma F(x)P_k(x)dx,$$

have on γ *(and hence*|| *in the closed region* $|x|\leqq 1$) *essentially the same con-*

* Presented to the Society, September 9, 1927; received by the editors in May, 1927.

† These Transactions, vol. 26 (1924), pp. 155–170. We shall refer to this paper as I.

‡ I, p. 159.

§ See below, § 2.

|| A convergent series of constant terms dominates the term-by-term difference of series (3) and (4) for $|x|=1$ and hence for $|x|\leqq 1$.

vergence properties, in the sense that their term-by-term difference approaches uniformly and absolutely the sum zero. In particular if $F(x)$ is continuous for $|x|\leqq 1$, analytic for $|x|<1$, and satisfies a Lipschitz condition on γ, then the series (4) *converges uniformly to the sum $F(x)$ in the closed region $|x|\leqq 1$.*

In I this theorem was applied, after conformal transformation, to obtain the two other theorems mentioned, the first on the expansion of an analytic function in terms of polynomials, the second including the analogue of the Laurent series. In the present paper we treat (Part A) more in detail the analogy between the two series (3) and (4), considering arbitrary series of type (4), the analogue of Abel's theorem and its converse, convergence properties on circles other than γ, and the uniqueness of expansions. In Part B we apply these results to the case of polynomials belonging to a given region, and collect the main results of the paper in Theorem IX. We consider in particular the expansion of a discontinuous function, in Theorem XI. It is found that under certain conditions Gibbs's phenomenon occurs, precisely as for Fourier's series. In Part C we study the use of polynomial expansions in connection with multiply-connected regions, obtaining certain results on the boundary values of analytic functions.

A. Series of analytic functions

2. **Modification of proof of Theorem I.** The proof of Theorem I given in I is needlessly complicated. It is perhaps worth while to present in some detail a modification, for we shall need later certain inequalities obtained.

We apply the Lemma used in I, choosing the interval $0\leqq\phi\leqq 2\pi$ as the circle γ: $|x|=1$, using $x=e^{i\phi}$ on γ. The functions $\{u_n(\phi)\}$ and $\{U_n(\phi)\}$ are taken (modifying the argument of I, pp. 162–3) simply as

$$u_n(\phi)=\frac{x^n}{(2\pi)^{1/2}},\qquad U_n(\phi)=\frac{p_n(x)}{(2\pi)^{1/2}}\qquad (n=0,1,2,\cdots). \tag{5}$$

Thus we have immediately

$$c_{nk}=\int_0^{2\pi}(U_n-u_n)\bar{u}_k d\phi, \tag{6}$$

$$\sum_{k=0}^{\infty}c_{nk}\bar{c}_{nk}\leqq\int_0^{2\pi}(U_n-u_n)(\overline{U}_n-\bar{u}_k)d\phi\leqq\epsilon_n^2, \tag{7}$$

$$\begin{aligned} c_{nk}=\int_0^{2\pi}(U_n-u_n)\bar{u}_k d\phi &=\frac{1}{2\pi i}\int_{\gamma}(p_n(x)-x^n)\frac{dx}{x^{k+1}}\\ &=\frac{1}{2\pi i}\int_{\gamma'}(p_n(x)-x^n)\frac{dx}{x^{k+1}},\end{aligned}$$

$$(8) \qquad | c_{nk} | \leqq \frac{\epsilon_n}{(1+\epsilon)^k} \cdot$$

The function $V_k(\phi)$ of I is therefore given by the equation*

$$(9) \qquad V_k(\phi) = \sum_{i=0}^{\infty}(d_{ki} + \delta_{ki})u_i.$$

We have, however, the inequalities

$$(10) \qquad | d_{ki} + c_{ik} | \leqq \frac{p_i p_k}{1-p}, \qquad p_i^2 = \sum_{j=0}^{\infty} | c_{ij} |^2, \qquad p^2 = \sum_{i,j=0}^{\infty} | c_{ij} |^2,$$

$$(11) \qquad \sum_{i=0}^{\infty} | d_{ki} | \ll \sum_{i=0}^{\infty} | d_{ki} + c_{ik} | + \sum_{i=0}^{\infty} | c_{ik} | \, .$$

We define the functions $P_k(x)$ so as to make the two following series identical:

$$\sum_{k=0}^{\infty} c_k p_k(x), \qquad c_k = \int_\gamma F(x) P_k(x) dx,$$

$$(12) \qquad \sum_{k=0}^{\infty} b_k U_k(\phi), \qquad b_k = \int_\gamma F(x) \overline{V}_k(\phi) d\phi.$$

That is, we set

$$c_k = \frac{b_k}{(2\pi)^{1/2}}, \quad P_k(x) = \frac{1}{(2\pi)^{1/2}} \overline{V}_k(\phi) \frac{d\phi}{dx} \cdot$$

We have of course

$$x = e^{i\phi}, \quad dx = ie^{i\phi} d\phi, \quad \frac{d\phi}{dx} = \frac{1}{ix} \cdot$$

It follows, then, directly from (9) and (5) that the functions $P_k(x)$ are continuous for $|x| \geqq 1$, analytic for $|x| > 1$, and zero at infinity. Moreover, if the series $\sum \epsilon_k$ is dominated by a convergent geometric series, then the functions $P_k(x)$ are analytic likewise for $|x| = 1$.† In fact, the series (11) is also dominated by a convergent geometric series, by virtue of (7):

$$p_i \leqq \epsilon_i,$$

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* See Walsh, these Transactions, vol. 22 (1921), p. 234, where the Lemma used in I is proved, and inequalities (10) and (11) likewise derived.

† The writer withdraws the statement in I, pp. 159, 163, that the functions $P_k(x)$ are analytic *on* γ, when the ϵ_k are not further restricted. Thus in the proof of I, Theorem I, we choose the ϵ_k so that the series $\sum \epsilon_k$ is dominated by a convergent geometric series.

and by virtue of (8). Then the series (9), when conjugate complex quantities are taken, is a Laurent series whose coefficients are dominated by a convergent geometric series, so $\overline{V}_k(x)$, and hence also $P_k(x)$, is analytic for $|x|=1$.

3. **Development of continuous functions on γ.** If the function $F(x)$ is continuous for $|x|\leqq 1$ and analytic for $|x|<1$, then the Taylor development of $F(x)$ about the origin converges, when summed by the method of Cesàro, uniformly for $|x|\leqq 1$ to the value $F(x)$. In fact the Taylor development is on γ precisely the Fourier development of $F(x)$, which when summed as described converges uniformly on γ to the value $F(x)$, hence uniformly on and within γ to the value $F(x)$. The Taylor series itself converges for $|x|<1$, by the usual inequalities for the coefficients of a power series, and hence converges to the value $F(x)$, because in case of a convergent series the sum assigned by the Cesàro summation process is the sum of the series.

Application of this remark yields, if we remember that series (3) and (4) have essentially the same convergence properties in the entire closed region $|x|\leqq 1$,

THEOREM II. *If $F(x)$ is continuous for $|x|\leqq 1$ and analytic for $|x|<1$, then the series* (4) *converges uniformly to the sum $F(x)$ in any closed region $|x|\leqq|x_0|<1$, and the sequence formed from* (4) *by the Cesàro summation method converges uniformly for $|x|\leqq 1$, to the sum $F(x)$.*

We turn now from the consideration of series (4) arising from functions $F(x)$ given on γ, to the consideration of series of the form

$$\sum_{k=0}^{\infty} g_k p_k(x) \tag{13}$$

with arbitrary coefficients g_k.

4. **Convergence of arbitrary series** (13). If no further restriction is placed on the quantities ϵ_k than in Theorem I, it is not true that the convergence of (13) for $x=x_0$ enables us to conclude the convergence of (13) for all values of x such that $|x|<|x_0|$. Let us set, in fact,

$$p_0(x) = 1, \quad p_k(x) = x^k - \delta^k, \quad k > 0,$$

where δ is positive and so small that for $\epsilon_0=0$, $\epsilon_k=\delta^k$, $k>0$, the required conditions on ϵ_k are fulfilled. Then every series (13) converges for $x=\delta$, yet need not converge for every x such that $|x|<\delta$. Indeed, under this same definition for $p_k(x)$, every series

$$\sum_{k=0}^{\infty} g_{2^k} p_{2^k}(x)$$

converges whenever $x=\omega\delta$, $\omega^{2^n}=1$, n being integral. This series converges therefore on a point set everywhere dense on the circle $|x|=\delta$, yet does not necessarily converge for $x=0$.

Under suitable restrictions on the ϵ_k we can prove the result for series (13) which is analogous to the well known result for Taylor's series:

THEOREM III. *If the series* (13) *converges for* $x=x_0$, *where* $|x_0|\leqq 1+\epsilon$, *and if the series* $\sum_{k=0}^{\infty}\epsilon_k t^k$ *converges for every (finite) value of* t, *then the series* (13) *converges for all values of* x *such that* $|x|<|x_0|$, *and the convergence is uniform for all values of* x *such that* $|x|\leqq|x_1|<|x_0|$.

We naturally assume $x_0\neq 0$; the contrary case is without content. We prove actually a stronger theorem than that stated, for we use not the convergence of (13) for $x=x_0$ but merely the boundedness of the terms of the series.

The inequality

$$| p_k(x_0) - x_0^k | \leqq \epsilon_k$$

gives at once the double inequality

$$1 - \frac{\epsilon_k}{|x_0^k|} \leqq \frac{|p_k(x_0)|}{|x_0^k|} \leqq 1 + \frac{\epsilon_k}{|x_0^k|}\,.$$

But we have $\lim_{k\to\infty} \epsilon_k/|x_0|=0$, and hence $\lim_{k\to\infty} |p_k(x_0)|/|x_0|=1$. Therefore if the quantities $g_k p_k(x_0)$ are uniformly bounded, so also are the quantities $g_k x_0$, and conversely. From the boundedness of the $g_k x_0$:

$$| g_k x_0^k | < M,$$

follows the inequality

$$| g_k | < \frac{M}{|x_0^k|}\,.$$

We now make use of the inequality

$$| p_k(x) | \leqq |x|^k + \epsilon_k,$$

or

$$\sum_{k=0}^{\infty} | g_k p_k(x) | \ll \sum_{k=0}^{\infty} | g_k | \cdot | x |^k + \sum_{k=0}^{\infty} | g_k | \epsilon_k.$$

The first series on the right converges uniformly for $|x|\leqq|x_1|$, since the individual terms of that series are bounded for $x=x_0$. The second series on the right, which does not contain x, converges. Hence the series on the left converges uniformly for $|x|\leqq|x_1|$, and Theorem III is established.

The argument just given includes practically a proof of the fact that for points x such that $|x|\leqq 1+\epsilon$, the two series

$$\sum_{k=0}^{\infty} g_k x^k \text{ and } \sum_{k=0}^{\infty} g_k p_k(x)$$

have the same points of convergence, of absolute convergence, of divergence, of summability, and the same regions (or point sets) of uniform convergence. The only exception here occurs if the Taylor series converges only at the point $x=0$. For if there exists a single value of x, say $x_0 \neq 0$, of the kind considered so that either of these two series converges, we have

$$| g_k | < \frac{M}{| x_0^k |} .$$

It follows that for all values of x, $|x| \leqq 1+\epsilon$, we have uniformly

$$| g_k p_k(x) - g_k x^k | \leqq \frac{M\epsilon_k}{| x_0^k |} .$$

The series $\sum_{k=0}^{\infty} M\epsilon_k / |x_0^k|$ converges and does not contain x, so the statement is proved.

There is no exceptional case here, even if the Taylor series converges only for $x=0$, provided $p_k(0)=0$, $k=1, 2, \cdots$; compare Theorems VII and IX below.

We can state now two interesting results of this discussion; the first result gives the radius of convergence in terms of the coefficients. Here and in the remainder of the paper we assume, unless otherwise stated, the series $\sum \epsilon_k t^k$ to converge for all values of t.

THEOREM IV. *If we set*

$$\overline{\lim_{k\to\infty}} | g_k |^{1/k} = \frac{1}{\rho},$$

then if $\rho \leqq 1+\epsilon$, *series* (13) *converges for* $|x| < \rho$ *and diverges for* $1+\epsilon > |x| > \rho$; *if* $\rho > 1+\epsilon$, *series* (13) *converges for* $|x| \leqq 1+\epsilon$.

The generalized theorem of Abel yields its analogue for the series (13):

THEOREM V. *If the series* (13) *converges for the value* $x=x_1$, *where* $|x_1| \leqq 1+\epsilon$, *then this series converges uniformly in the closed region bounded by two arbitrary line segments terminating at the point* x_1 *and by an arc of the circle* $|x| = |x_2| < |x_1|$.

If (13) *converges uniformly on an arc* x_1x_2 *of the circle* $|x| = |x_1|$, *then this series converges uniformly in the closed region bounded by this circular arc, by two arbitrary line segments lying in the region* $|x| \leqq |x_1|$ *and terminated respectively by* x_1 *and* x_2, *and by an arc of a circle* $|x| = |x_3| < |x_1|$.

5. **Analogue of Tauber's theorem.** For series (13) we can give likewise a converse (not exact) of Abel's Theorem:

THEOREM VI. *If the series* (13) *is such that* $\lim_{k\to\infty} g_k/k = 0$, *and if for radial approach* to the point* x_1 *on* γ *we have*

$$\lim_{x\to x_1} f(x) = g,$$

where $f(x)$ *denotes the value of the* (*convergent for* $|x| < 1$) *series* (13), *then we have also*

$$\sum_{k=0}^{\infty} g_k p_k(x_1) = g.$$

If we define the numbers b_k by the relations

$$b_k = (2\pi)^{1/2} g_k,$$

and then set

$$a_k = b_k + c_{0k}b_0 + c_{1k}b_1 + c_{2k}b_2 + \cdots \qquad (k = 0,1,2,\cdots), \tag{14}$$

we find by the use of the Schwarz inequality in conjunction with (8),

$$|a_k - b_k| \leqq \frac{\left(\sum_{i=0}^{\infty} |b_i|^2\right)^{1/2}}{(1+\epsilon)^k}. \tag{15}$$

By our hypothesis on the g_k, the series $\sum_{i=0}^{\infty} |b_i|^2$ converges. The a_k defined by (14) are such that $\lim_{k\to\infty} a_k/k = 0$, by (15). These numbers a_k are such that $\sum_{k=0}^{\infty} |a_k|^2$ converges, hence, by the Riesz-Fischer theorem, there exists a function $F(x)$ defined on γ, integrable and with an integrable square, whose coefficients with respect to the normal orthogonal system $\{u_k\}$ used in §2 are the numbers a_k. The numbers b_k, subjected to the condition that $\sum_{k=0}^{\infty} |b_k|^2$ should converge, are uniquely determined by (14),† and hence the numbers $g_k = b_k/(2\pi)^{1/2}$ are the coefficients of $F(x)$ in its expansion (4). Thus (3) and (13) have essentially the same convergence properties in and on γ.

By Tauber's theorem,‡ we have, since $\lim_{k\to\infty} a_k/k = 0$,

$$\sum_{k=0}^{\infty} a_k x_1^k = g,$$

* The result holds also for approach in various other ways. See for example Landau, *Ergebnisse der Funktionentheorie*, Berlin, 1916, Kap. III. Compare our application of Theorem VI in Theorem IX.

† See the reference in § 2 to the proof of the Lemma used in I.

‡ See Landau, loc. cit.

and thus we have as well

$$\sum_{k=0}^{\infty} g_k p_k(x_1) = g,$$

and the theorem is established.

Many other results similar to Theorem VI, analogues of results for Taylor's series, might be established. We choose, however, to treat the equivalence of series (3) and (4) on circles other than γ.

6. **Properties of series on circles other than γ.** We prove the following theorem:

THEOREM VII. *If $p_k(x)$ has at least a k-fold root at the origin, then an arbitrary function $F(x)$ integrable and with an integrable square on the circle Γ: $|x|=\mu<1+\epsilon$ can be formally expanded in a series of type* (4), *where the coefficients are found by integration over Γ. This series* (4) *and the Taylor development (formal) of $F(x)$ have on and within the circle Γ the same convergence properties, in the sense that their term-by-term difference converges absolutely and uniformly on Γ and in its interior to the sum zero.*

We perform the substitution $z=x/\mu$, $x=\mu z$, so as to apply Theorem I directly to the unit circle in the z-plane. We require for application of Theorem I the inequality

$$\left|\frac{p_k(\mu z)}{\mu^k} - z^k\right| \leqq \epsilon_k \text{ for all } |z| \leqq 1 + \epsilon', \quad \epsilon' > 0. \tag{16}$$

Expansion of the function $F(\mu z)$ on the circle $|z|=1$ in terms of the functions $p_k(\mu z)/\mu^k$, which approximate to the functions z^k, will yield of course a formal expansion of $F(x)$ on the circle Γ in terms of the functions $p_k(x)$. The Taylor expansion of $F(\mu z)$, a power series in z, transforms into a Taylor expansion of $F(x)$, a power series in x.

Our original inequality

$$|p_k(x) - x^k| \leqq \epsilon_k, \quad |x| \leqq 1 + \epsilon,$$

may be written

$$\left|\frac{p_k(x)}{x^k} - 1\right| \leqq \frac{\epsilon_k}{|x^k|}, \qquad x \neq 0.$$

But the function $(p_k(x)/x^k)-1$ is analytic without exception for $|x|\leqq 1+\epsilon$, when properly defined for $x=0$, and its greatest absolute value in that closed region is taken on for $|x|=1+\epsilon$. Thus we have

$$\left|\frac{p_k(x)}{x^k} - 1\right| \leqq \frac{\epsilon_k}{(1+\epsilon)^k}, \quad |x| \leqq 1 + \epsilon.$$

Transformation to the z-plane gives the equivalent inequalities

$$\left|\frac{p_k(\mu z)}{\mu^k z^k} - 1\right| \leqq \frac{\epsilon_k}{(1+\epsilon)^k}, \qquad \mu|z| \leqq 1+\epsilon,$$

$$\left|\frac{p_k(\mu z)}{\mu^k} - z^k\right| \leqq \frac{\epsilon_k |z^k|}{(1+\epsilon)^k}, \qquad \mu|z| \leqq 1+\epsilon. \tag{17}$$

The right-hand member of (17) is not greater than ϵ_k provided we restrict z as follows:

$$|z| \leqq \frac{1+\epsilon}{\mu} \text{ if } \mu > 1, \qquad |z| \leqq 1+\epsilon \text{ if } \mu \leqq 1. \tag{18}$$

The upper limits of z in (18) are both greater than unity, so (17) yields directly (16), and Theorem VII is completely established.

The proof of Theorem VII has not assumed any restriction on the quantities ϵ_k beyond that of Theorem I. In fact, the condition that $p_k(x)$ should have at least a k-fold root at the origin is a very favorable one with reference to successive approximations and equivalence of expansions. With the conditions imposed, the requirements on the ϵ_k of Theorem I can be considerably lightened; we do not, however, carry out the details here.

We remark, too, that a result similar to Theorem VII is readily proved under the assumption that a convergent geometric series dominates the series $\sum \epsilon_k$, without the assumption that $p_k(x)$ has at least a k-fold root at the origin; but here there is in general a lower limit *greater than zero* on the radius μ of the circle Γ. We omit the proof of this remark.

The following theorem is by no means the most general result of its kind that can be easily established:

THEOREM VIII. *If $p_k(x)$ has at least a k-fold root at the origin, and if the series $\sum \epsilon_k t^k$ converges for every t, then the expansion of type* (4) *of any function $F(x)$ analytic at the origin is unique. The functions $P_k(x)$ of Theorem* I *are analytic over the entire plane except at the origin.*

If $F(x)$ is analytic on and within γ, there cannot exist two distinct expansions of $F(x)$ of the form

$$F(x) = \sum_{k=0}^{\infty} c_k p_k(x), \qquad F(x) = \sum_{k=0}^{\infty} g_k p_k(x) \tag{19}$$

both of which converge uniformly on γ. For multiplication of these series through by $P_k(x)\,dx$ and integration term by term over γ gives by (2) the equality of c_k and g_k.

We return to the more general situation of Theorem VIII. If two series of the form (19) both converge at even a single point for which $|x| \leqq 1+\epsilon$, they converge uniformly on and within some circle Γ' whose center is the origin. By the remark just made concerning uniqueness of expansions, and by the proof of Theorem VII, the two expansions are identical if they represent the same function on any circle whatever whose center is the origin. If the two series represent $F(x)$ in a region lying interior to the circle $|x| = 1+\epsilon$, they both represent that function throughout their entire regions of convergence interior to the circle $|x| = 1+\epsilon$.

The analyticity of the functions $P_k(x)$ of Theorem I over the entire plane except at the origin follows from (9) and (11) as used in §2, with the new properties of the series $\sum \epsilon_k$. Compare also Theorem IXa, which does not use those new properties. Under the present hypothesis, then, the integrals which appear in (4) can be taken over any rectifiable Jordan curve which lies interior to γ and in whose interior the origin lies, provided the function $F(x)$ is analytic for $|x| \leqq 1$. The functions $P_k(x)$ which arise in Theorem I for the circle γ, and the functions $P_k(x)$ which arise in the proof of Theorem VII by application of Theorem I to the transform in the z-plane of the circle Γ are identical; this can be verified by making the change of variable in all the formulas involved.

B. Series of polynomials

7. **Application of results of A.** We now apply Theorem I and the theorems which have just been proved in connection with it, deriving results as in I (p. 163 et seq.) for expansions of arbitrary functions in terms of polynomials. We choose the quantities ϵ_k to satisfy the requirements of Theorem I and also so that $\sum \epsilon_k t^k$ converges for every t.* The polynomial $p_k(z)$ is to be chosen so as to have a k-fold root at the origin; this choice is possible; compare I, p. 164, or Theorem X below. Then we have

Theorem IX. *In the plane of the complex variable z let C be a simple closed finite analytic curve*† *which includes in its interior the origin. Then*

* See also the condition of I, p. 164, and its application in the proof of Theorem IXa.

† That is to say, a curve whose points can be put into one-to-one (regular-) analytic correspondence with the points of a circle. It is then a classical theorem in the study of conformal mapping that the region interior to C can be mapped on the interior of a circle so that the mapping is one-to-one and conformal in the *closed* regions considered, therefore one-to-one and conformal in larger regions including those closed regions in their interiors. See Picard, *Traité d'Analyse*, II, Paris, 1893, pp. 272, 276, or Bieberbach, *Einführung in die konforme Abbildung*, Berlin, 1913, p. 120.

From this theorem it follows at once, in the notation of I or of Theorem IX, that for points z_1 and z_2 on C, the quotients $(z_1-z_2)/(\phi(z_1)-\phi(z_2))$ and $(\phi(z_1)-\phi(z_2))/(z_1-z_2)$ are uniformly bounded. Hence a function which satisfies a Lipschitz condition on C corresponds to a function which satisfies a Lipschitz condition on the circle γ, and conversely.

the interior of C can be mapped one-to-one and conformally on the interior of the unit circle γ *in the x-plane by some transformation* $x=\phi(z)$, $z=\psi(x)$, *where* $\phi(0)=0$, *and the transformation will be one-to-one and conformal for the mapping of the closed interior of* $C_{1+\epsilon}$, *an analytic Jordan curve in whose interior C lies, onto the closed interior of the circle* $|x|=1+\epsilon$, $\epsilon>0$. *In general we denote by* C_ρ *the transform of the circle* $|x|=\rho$, *where* $0<\rho<1+\epsilon$.

Then there exists a set $\{p_k(z)\}$ *of polynomials in z and a set of functions* $\{s_k(z)\}$ *analytic at every point of the extended plane except the origin, zero at infinity, and such that*

$$\int_{C_\rho} s_k(z)p_i(z)dz = \begin{cases} 0, & i \neq k, \\ 1, & i = k. \end{cases}$$

If the function $F(z)$ *is analytic interior to* C_ρ, *continuous in the corresponding closed region, and satisfies a Lipschitz condition on* C_ρ, *then the series*

$$\sum_{k=0}^{\infty} a_k p_k(z), \quad a_k = \int_{C_\rho} F(z)s_k(z)dz, \tag{20}$$

converges uniformly in this closed region to the value $F(z)$. *If* $F(z)$ *is required merely to be analytic interior to* C_ρ *and continuous in the corresponding closed region, then* (20) *converges uniformly to the value* $F(z)$ *interior to an arbitrary curve* $C_{\rho'}$, $\rho'<\rho$, *and when summed by the method of Cesàro,* (20) *converges uniformly to the value* $F(z)$ *in the closed region bounded by* C_ρ.

If $F(z)$ *is an arbitrary function defined on* C_ρ, *integrable and with an integrable square, and if the condition**

$$\int_{C_\rho} F(z)z^k dz = 0 \qquad (k = 0,1,2,\cdots),$$

is satisfied, then the two series

$$\sum_{k=0}^{\infty} a_k' x^k, \quad a_k' = \frac{1}{2\pi i}\int_{|x|=\rho} F[\psi(x)]x^{-k-1}dx,$$

and (20) *transformed by* $z=\psi(x)$ *have essentially the same convergence properties on and within the circle* $|x|=\rho$, *in the sense that their term-by-term difference converges absolutely and uniformly to the sum zero for* $|x|\leqq\rho$.

---

* No condition is necessary here if we use

$$a = \int_{|x|=\rho} F[\psi(x)]P_k(x)dx$$

instead of (20).

An arbitrary series of the form

$$\sum_{k=0}^{\infty} g_k p_k(x) \tag{21}$$

which converges for a single point z on C_ρ, converges uniformly interior to $C_{\rho'}$, if $\rho' < \rho$. If (21) *diverges for a point z on C_ρ that series diverges for all points z exterior to C_ρ and interior to $C_{1+\epsilon}$. If in general we set*

$$\limsup_{k\to\infty} |g_k|^{1/k} = \frac{1}{\rho}$$

then if $\rho \leqq 1+\epsilon$, series (21) *converges for z interior to C_ρ and diverges for z exterior to C_ρ but interior to $C_{1+\epsilon}$; if $\rho > 1+\epsilon$, series* (21) *converges for z on or interior to $C_{1+\epsilon}$. If $0 < \rho < 1+\epsilon$, some singular point of the function represented by the series lies on the curve C_ρ.*

If (21) *converges for the value $z = z_1$ on C_ρ, then this series converges uniformly in the closed region bounded by two arbitrary line segments terminating in z_1, and by an arc of a curve $C_{\rho'}$, where $\rho' < \rho$. If* (21) *converges uniformly on an arc $z_1 z_2$ of the curve C_ρ then this series converges uniformly in the closed region bounded by that arc, by two arbitrary line segments whose interiors are interior to C_ρ and which are terminated respectively by z_1 and z_2, and by an arc of the curve $C_{\rho'}$, where $\rho' < \rho$.*

If (21) *is such that $\lim_{k\to\infty} g_k(k\rho^k)^{-1} = 0$, and if for approach along the normal* to C_ρ to the point z_1 on C_ρ we have*

$$\lim_{z\to z_1} f(z) = g,$$

where $f(z)$ denotes the value of the (convergent for z interior to C_ρ) series (21), *then we have also*

$$\sum_{k=0}^{\infty} g_k p_k(z_1) = g.$$

An arbitrary series (21), *convergent for a single value of z interior to $C_{1+\epsilon}$ and not the origin, is the unique expansion of form* (20) *of some function $F(z)$ analytic on and within some curve C_ρ.*

The only part of this theorem not a direct result of our previous theorems is the fact that $s_k(z)$ is analytic over the entire z-plane except at the origin. This should give the reader no difficulty, using Theorem VIII and the method of I, p. 165; compare also §9.

* A more general theorem might easily be announced; see the footnote to Theorem VI. Here we do not apply Theorem VI directly, but the more general theorem suggested in connection with Theorem VI.

It will be noticed that Theorem IX does not mention convergence of the series (20) or (21) outside of $C_{1+\epsilon}$. The reason for this omission will become clearer after we have proved a general theorem on approximation.

8. **A general theorem on approximation.** We prove a much more general theorem than necessary for our immediate purposes:*

THEOREM X. *If the function $f(z)$, defined on the bounded point set S, can be approximated on that point set as closely as desired by a polynomial in z, and if there be given any p points $z_1, z_2, \cdots, z_p$ of S together with an arbitrary $\epsilon > 0$, then there exists a polynomial $p(z)$ such that*

$$|p(z) - f(z)| \leqq \epsilon, \quad z \text{ on } S,$$

and

$$p(z_i) = f(z_i) \qquad (i = 1, 2, \cdots, p).$$

We prove Theorem X by means of Lagrange's Interpolation Formula, and find it convenient first to prove the following

LEMMA. *If $z_1, z_2, \cdots, z_p$, R are considered fixed, if we have*

$$|G_k| \leqq \eta \qquad (k = 1, 2, \cdots, p),$$

and if $G(z)$ denotes the polynomial defined by Lagrange's Interpolation Formula which takes on the values G_k in the points z_k, $k = 1, 2, \cdots, p$, then there exists a constant M independent of η so that we have

$$|G(z)| \leqq M\eta \text{ for all } z, \ |z| \leqq R.$$

For simplicity we take R so large that $|z_k| \leqq R$, $k = 1, 2, \cdots, p$. The Lagrange Formula is

$$G(z) = \sum_{\nu=1}^{p} G_\nu \frac{(z - z_1) \cdots (z - z_{\nu-1})(z - z_{\nu+1}) \cdots (z - z_p)}{(z_\nu - z_1) \cdots (z_\nu - z_{\nu-1})(z_\nu - z_{\nu+1}) \cdots (z_\nu - z_p)},$$

so the Lemma is obvious if we merely set

$$M = \sum_{\nu=1}^{p} \frac{(2R)^{p-1}}{|z_\nu - z_1| \cdots |z_\nu - z_{\nu-1}| \cdot |z_\nu - z_{\nu+1}| \cdots |z_\nu - z_p|}.$$

* It seems inconceivable that this theorem is not already in the literature, but the writer has been unable to find a reference to it. The corresponding theorem for approximation by trigonometric polynomials is given by D. Jackson, Bulletin of the American Mathematical Society, vol. 32 (1926), pp. 259–262.

Theorem X holds of course for approximation of real functions by means of real polynomials, and can be extended (1) by requiring the agreement of certain derivatives of the approximating polynomial with the corresponding derivatives of the given function, (2) by assigning as the values of the polynomial (and derivatives) not the exact but values near to the values of the function (and derivatives), (3) by noticing that for points z_k off S arbitrary values $f(z_k)$ may be assigned.

Theorem X follows easily now. Choose R so large that all points of S lie in the circle $|z| \leqq R$. Choose a polynomial $q(z)$ (which exists by hypothesis) so that we have

$$| q(z) - f(z) | \leqq \frac{\epsilon}{1 + M}, \quad z \text{ on } S.$$

For the polynomial $G(z)$ we assign the values

$$G(z_k) = q(z_k) - (fz_k) \qquad (k = 1, 2, \cdots, p),$$

so that we have by the Lemma

$$| G(z) | \leqq \frac{M\epsilon}{1 + M}, \quad z \text{ on } S.$$

Then the polynomial

$$p(z) = q(z) - G(z)$$

satisfies all the requirements of Theorem X.

The polynomials $p_k(z)$ of Theorem IX are polynomials which uniformly approximate to the functions $[\phi(z)]^k$ respectively on and within $C_{1+\epsilon}$. By a classical theorem due to Runge, these polynomials may be subjected to the auxiliary condition of uniformly approximating other analytic functions —let us say constants—in arbitrary non-intersecting closed Jordan regions outside of $C_{1+\epsilon}$. In particular we may by Theorem X require that the polynomials $p_k(z)$ shall actually take on arbitrarily preassigned values at an arbitrary number of points exterior to $C_{1+\epsilon}$. Thus we may choose (1) the value zero for the points $z_1, z_2, \cdots, z_p$ (independent of k), in which case all series of the form (21) converge at those points, or we may choose (2)

$$p_k(z_i) = k! \qquad (i = 1, 2, \cdots, p; \quad k = 1, 2, 3, \cdots),$$

in which case no series (21) not convergent throughout the interior of $C_{1+\epsilon}$ converges at the points z_i. It is because of this difference in behavior that Theorem IX omits mention of the convergence or divergence of (21) outside of $C_{1+\epsilon}$.

9. **Further properties of expansions.** One interesting property of series (20) has not yet been mentioned, which brings out still more clearly the analogy with Taylor's series:

THEOREM IXa. *The coefficients a_k in* (20) *can be written in the form*

$$a_k = A_0^{(k)}F(0) + A_1^{(k)}F'(0) + \cdots + A_k^{(k)}F^{(k)}(0),$$

where $A_i^{(k)}$ is a constant independent of $F(z)$, and where $F^{(i)}(0)$ indicates the ith derivative of $F(z)$ at the origin.

Differentiation of (20), with insertion of the value $z=0$, yields

$$
\begin{aligned}
F(0) &= a_0 p_0(0), \\
F'(0) &= \qquad a_1 p_1'(0), \\
F''(0) &= \qquad a_1 p_1''(0) + a_2 p_2''(0), \\
F'''(0) &= \qquad a_1 p_1'''(0) + a_2 p_2'''(0) + a_3 p_3'''(0), \\
&\cdots\cdots\cdots\cdots
\end{aligned}
$$

Here we use the fact that $p_0(z)$ is constant, and that $p_k(z)$ has at least a k-fold root at the origin and hence (I, p. 164), having precisely k roots interior to $C_{1+\epsilon}$, has precisely a k-fold root at the origin. This system of equations is therefore such that $p_k^{(k)}(0)$ is always different from zero, $k=0, 1, 2, \cdots$, and hence the system can be solved for the coefficients a_k linearly in terms of the $F^{(i)}(0)$.

As an application of Theorem IXa, it may be noticed that $s_k(z)$ can be written in the form

$$
S_0(z) = \frac{B_1^{(0)}}{z}, \; S_k(z) = \frac{B_2^{(k)}}{z^2} + \frac{B_3^{(k)}}{z^3} + \cdots + \frac{B_{k+1}^{(k)}}{z^{k+1}}, \; k > 0\,;
$$

this follows directly from the integral formula for the derivatives of $F(z)$.

One may consider in some detail the expansion of an arbitrary function $\Phi(z)$, analytic on and interior to C, in terms not of the polynomials $p_k(z)$ but in terms of their derivatives $p_k'(z)$. Let $F(z)$ be any integral of $\Phi(z)$, so that we have for z on and interior to C,

$$
F(z) = a_0 p_0(z) + a_1 p_1(z) + a_2 p_2(z) + \cdots, \qquad a_k = \int_C F(z) s_k(z) dz,
$$

$$
F'(z) = \Phi(z) = a_1 p_1'(z) + a_2 p_2'(z) + \cdots.
$$

The term $a_0 p_0'(z)$ is here omitted, for $p_0(z)$ is a constant.

The integral

$$
\int_C s_k(z) dz, \quad k > 0,
$$

is equal to zero, for this integral may be written

$$
\frac{1}{p_0(z)} \int_C s_k(z) p_0(z) dz,
$$

known to vanish by Theorem IX. Hence the indefinite integral $\sigma_k(z)$ of $s_k(z)$ is single-valued in and on C.

Let us integrate

$$a_k = \int_C F(z) s_k(z) dz, \quad k > 0,$$

by parts, $\int u dv = uv - \int v du$, setting $u = F(z)$, $dv = s_k(z)dz$. We find

$$a_k = -\int_C \Phi(z)\sigma_k(z) dz.$$

As is to be expected, we find also by partial integration

$$\int_C \sigma_k(z) p_i'(z) dz = -\int_C s_k(z) p_i(z) dz = -\delta_{ik}.$$

That is, there exists a set of functions $\{-\sigma_k(z)\}$ such that the two sets $\{p_k'(z)\}$ and $\{-\sigma_k(z)\}$ are biorthogonal. An arbitrary function $\Phi(z)$ analytic on and within C can be expanded in the series

$$\Phi(z) = a_1 p_1'(z) + a_2 p_2'(z) + \cdots, \quad a_k = -\int_C \Phi(z)\sigma_k(z) dz,$$

which converges uniformly on and within C.

Both this remark on the derived functions and Theorem IXa can be applied in the x-plane to the series (4) under the hypothesis of Theorem VIII.

10. **Expansion of discontinuous functions.** There are considered in I not merely series such as (20), but likewise series in polynomials $q_k(z)$ in the reciprocal of z.* These series are used in I to expand arbitrary functions satisfying a Lipschitz condition on C. In order to study the expansion of *discontinuous* functions in such series, we investigate the function or functions represented by Cauchy's Integral

$$F(z) = \frac{1}{2\pi i} \int_C \frac{f(t)dt}{t - z},$$

where the given function $f(t)$ is discontinuous. We shall suppose C to be the same curve previously considered, although the discussion holds under much broader conditions.

A particularly simple kind of discontinuity, that of a finite jump, is typified by the function $f(t) = \log t$, where we choose as that branch of the

* We notice that the argument used in I, p. 166, to prove $b_0 = 0$ can be somewhat shortened. We choose, in fact, $q_0(z) = 1$, $q_k(\infty) = 0$ for $k > 0$. Then in the expansion of $f_2(z)$:

$$f_2(z) = \sum_{k=0}^{\infty} b_k q_k(z),$$

it is obvious that $b_0 = 0$ when $f_2(z) = 0$ for $z = \infty$.

function log t the branch which is real and positive for the smallest real positive value of t on C, say t_0. Consider in general the plane cut along the line $0t_0$, and from t_0 to infinity along a curve exterior to C. The determination of the branch of log t considered is then made by the use of continuity in the cut plane. The function $f(t)$ is continuous on C except for a finite jump at t_0 of magnitude $2\pi i$.

We evaluate the integral

$$F(z) = \frac{1}{2\pi i}\int_C \frac{\log t}{t-z}\,dt$$

by partial integration, $\int u\,dv = uv - \int v\,du$, setting $u = \log t$, $dv = dt/(t-z)$. We find

$$F(z) = \frac{1}{2\pi i}\log t\cdot\log(t-z)\Big|_C - \frac{1}{2\pi i}\int_C \frac{\log(t-z)}{t}\,dt.$$

The first term in the right-hand member has the value $2\pi i + \log t_0 + \log(t_0 - z)$, or $\log(t_0 - z)$, according as z lies interior or exterior to C. The proper determination of $\log(t-z)$ is to be found by continuity, moving t along C until it coincides with t_0, then by moving z, not crossing the cut, until z coincides with the origin. The second term in the right-hand member has a zero derivative with respect to z, if z lies interior to C, as is seen by direct computation. The value of the integral, z interior to C, is therefore a constant equal to the value for $z=0$:

$$-\frac{1}{2\pi i}\int_C \frac{\log t}{t}\,dt = -\frac{1}{4\pi i}\log^2 t\Big|_C = -\pi i - \log t_0.$$

The value of this same integral

$$-\frac{1}{2\pi i}\int_C \frac{\log(t-z)}{t}\,dt$$

when z lies exterior to C is, by Cauchy's Formula, $-\log(-z)$. We have finally, therefore,

$$f_1(z) \equiv F(z) = \pi i + \log(t_0 - z), \quad z \text{ interior to } C,$$

$$f_2(z) \equiv -F(z) = -\pi i - \log(t_0 - z) + \log z, \quad z \text{ exterior to } C.$$

As a check we have $f(z) = f_1(z) + f_2(z)$ when z lies on C (except in case $z = t_0$, where the functions $f_1(z)$ and $f_2(z)$ are, strictly speaking, not defined), as we should have by the results of Plemelj.* The function $f_1(z)$ is analytic on

* See I, p. 167. The validity of the equation $f(z) = f_1(z) + f_2(z)$ is dependent, provided $f(z)$ satisfies certain large conditions of integrability, merely on the behavior of the function $f(z)$ in the neighborhood of the point z considered; the satisfaction of a Lipschitz condition in such a neighborhood is sufficient for the validity of the equation.

and interior to C except at the single point t_0; the function $f_2(z)$ is analytic on and exterior to C except at t_0 and vanishes at infinity; both functions are integrable and have an integrable square on C. We notice too by direct computation

$$\int_C f_1(t)t^n dt = 0 \qquad (n = 0,1,2, \cdots),$$

$$\int_C f_2(t)t^n dt = 0 \quad (n = -1, -2, -3, \cdots),$$

from which follow the formulas (notation of I) for the coefficients in the expansion of $f_1(t)$ and $f_2(t)$:

$$(22) \qquad \begin{aligned} &\int_C f_1(t)t_k(t)dt = 0 \qquad (k = 1,2,3, \cdots), \\ &\int_C f_2(t)s_k(t)dt = 0 \qquad (k = 0,1,2, \cdots). \end{aligned}$$

We use here in proving (22) the fact that $t_k(t)$ is analytic on and within C, hence on C can be expressed as a uniformly convergent series of polynomials in t; likewise $s_k(t)$ is analytic on and exterior to C, hence on C can be expressed as a uniformly convergent series of polynomials in $1/t$ each without constant term. Such series may be integrated term by term, even after multiplication by $f_1(z)$ or $f_2(z)$.

The development of $f_1(z)$ on C in terms of the polynomials $p_k(z)$ has essentially the same convergence properties as the development of the function

$$\pi i + \log [t_0 - \psi(x)]$$

in a Fourier series (which is precisely the same as the development of the function in a Taylor or Laurent series) on the unit circle γ in the x-plane. The development of $f_2(z)$ on C in the polynomials $q_k(z)$ has essentially the same convergence properties as the development of

$$-\pi i - \log [t_0 - \psi_1(x)] + \log \psi_1(x)$$

in a Fourier (or Laurent) series on γ, where we may take the solutions $x = x_0$ of the two equations

$$\psi(x) = t_0, \quad \psi_1(x) = t_0$$

equal,* $\psi_1(x)$ being a mapping function for the exterior of γ onto the exterior of C, with correspondence of the points at infinity. These two functions

* Rotation of axes does not alter the convergence properties of a Taylor or Fourier development.

just considered in the x-plane are both integrable with an integrable square and on γ possess continuous derivatives except at the point x_0. The developments of the two functions converge therefore to the values of the respective functions except at x_0, and uniformly except in the neighborhood of x_0. In the neighborhood of the point x_0 the term-by-term sum of the two developments converges like the Fourier development of

$$\log \frac{t_0 - \psi(x)}{t_0 - \psi_1(x)} + \log \psi_1(x).$$

The latter term contains the only discontinuity, a finite jump of magnitude $2\pi i$. Gibbs's phenomenon therefore occurs in its characteristic form at this point x_0; the series converges to the value which is the arithmetic mean of the limits approached in the two directions on γ at x_0 by the function developed.

The Fourier development of $f_2(t)$ transformed by either $t=\psi(x)$ or $t=\psi_1(x)$ but interpreted for the same values of t has essentially the same convergence properties in the two cases.

The discussion we have given is not essentially dependent on the particular choice of t_0 made originally. We may therefore state

THEOREM XI. *If the function $F(z)$ satisfies a Lipschitz condition on C, and if the function*

$$f(z) = F(z) + k_1 \log z + k_2 \log z + \cdots + k_m \log z,$$

where each term $k_i \log z$ is continuous on C except at a single point z_i of C, $z_i \neq z_k$ $(i \neq k)$, $k_k \neq 0$,—be developed in a series (2) *as in* I (*p.* 156), *then the Fourier development of $f(z)$ on the unit circle $|x|=1$, where $z=\psi(x)$, and the series*

$$f(z) = a_0p_0(z) + [a_1p_1(z) + b_1q_1(z)] + [a_2p_2(z) + b_2q_2(z)] + \cdots \tag{23}$$

have essentially the same convergence properties on C. In particular* (23) *exhibits Gibbs's phenomenon at the points z_k precisely as does a Fourier series. On any closed arc of C containing no point z_k, the series*

$$a_0p_0(z) + a_1p_1(z) + a_2p_2(z) + \cdots \tag{24}$$

* The statement that two series *have the same convergence properties* is used in two senses in the literature, to indicate (1) that their term-by-term difference converges uniformly to the sum zero, or (2) that their term-by-term difference converges absolutely and uniformly to the sum zero. The present writer has hitherto consistently used the second of these two meanings, but in the present case implies (1) instead of (2). The treatment given here considers uniform convergence but not absolute convergence.

converges to the value $f_1(z)$ *and the series*

$$b_1q_1(z) + b_2q_2(z) + \cdots \tag{25}$$

converges uniformly to the value $f_2(z)$, *where*

$$f_1(z) \equiv \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z}$$

is analytic interior to C and continuous in the corresponding closed region except at the points z_k, *and*

$$f_2(z) \equiv \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z}$$

is analytic exterior to C, vanishes at infinity, and is continuous in the corresponding closed region except at the points z_k. *For* $z=z_k$ *the two series* (24) *and* (25) *diverge with infinite sum, whereas the series* (23) *converges and its sum is the arithmetic mean of the two limits approached by* $f(z)$ *as* z *moves in opposite senses on C and approaches* z_k. *If an arbitrary neighborhood of each of the points* z_k *is cut out of the closed interior of C, the series* (24) *converges uniformly to the value* $f_1(z)$ *in the remaining closed region. If an arbitrary neighborhood of each of the points* z_k *is cut out of the closed exterior of C, the series* (25) *converges uniformly to the value* $f_2(z)$ *in the remaining closed region.*

Theorem XI is proved under the hypothesis on the ϵ_k that $\sum \epsilon_k t^k$ converges for every t.

Actual formulas for $f_1(z)$ and $f_2(z)$ in terms of logarithms and the functions represented by the integral

$$\frac{1}{2\pi i}\int_C \frac{F(t)dt}{t-z}$$

can easily be written down. Of course, any function which is smooth except for a finite number of finite jumps can be put into the form of $f(z)$ of this theorem.

The conclusion of Theorem XI naturally holds for the formal Laurent development of a discontinuous function of the kind considered, if the curve C is a circle. In particular it is the divergence of the Taylor series for $\log(x-a)$ for $x=a$ that enables us to conclude the divergence of (24) and (25) for $z=z_k$.

C. Boundary values of an analytic function

11. A condition for analyticity. We now take up the study of the

boundary values of an analytic function, later for a multiply-connected region but first for a simply-connected region:*

THEOREM XII. *If the function $f(z)$ is continuous on the analytic Jordan curve C, and if we have*

$$\int_C f(z)z^n dz = 0 \qquad (n = 0, 1, 2, \cdots), \tag{26}$$

then there exists a function $F(z)$ analytic interior to C, continuous in the closed region which consists of C and its interior, and which coincides with $f(z)$ on C.

If the curve C is the unit circle $|z| = 1$, the theorem is surely true. In fact the formal Laurent development of $f(z)$ is of the form of a Taylor series, since by (26) the coefficients of the negative powers of z vanish:

$$f(z) \sim a_0 + a_1 z + a_2 z^2 + \cdots, \quad a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}} dz.$$

This development is precisely the formal Fourier development of $f(z)$ for $0 \leqq \phi \leqq 2\pi$ if we set $z = e^{i\phi}$. The Fourier development, when summed by the method of Cesàro, converges uniformly on C, since $f(z)$ is continuous. Each term of the corresponding sequence is analytic on and interior to C, hence the sum of the series is analytic interior to C, continuous in the corresponding closed region, and is equal to $f(z)$ on C. This completes the proof of the theorem when C is the unit circle.

If C is not the unit circle, we map the interior of C onto the interior of the unit circle γ in the x-plane, the transformation being as usual $x = \phi(z)$, $z = \psi(x)$. The function $[\phi(z)]^n \phi'(z)$ is analytic in and on C, hence on C can be (by Runge's theorem) uniformly expanded in a series of polynomials in z:

$$[\phi(z)]^n \phi'(z) = \pi_0(z) + \pi_1(z) + \pi_2(z) + \cdots, \; n \geqq 0.$$

This series converges uniformly on C even after multiplication term by term by the continuous function $f(z)$. Term-by-term integration of the new series thus formed yields, by virtue of (26),

$$\int_C f(z)[\phi(z)]^n \phi'(z) dz = 0 \qquad (n = 0, 1, 2, \cdots).$$

* In connection with this problem and the conditions derived, see F. and M. Riesz, *Comptes Rendus du Congrès* (1916) *des Mathématiciens Scandinaves*, Uppsala, 1920, pp. 27–44; Privaloff, *L'Intégrale de Cauchy*, Saratow, 1919; Kakeya, Tôhoku Mathematical Journal, vol. 5 (1914), pp. 40–44, as well as the references given in I, p. 167.

We have, then,

$$\int_\gamma f[\psi(x)]x^n dx = 0 \qquad (n = 0,1,2,\cdots),$$

so by the special case of the theorem already proved there exists a function analytic interior to γ, continuous in the corresponding closed region, and coinciding on γ with the function $f[\psi(x)]$. Transformation by the formula $x=\phi(z)$ gives the required function in the z-plane.

Conditions (26), it may be remarked, are all independent of each other and none of them may be omitted. If all of those conditions except a finite number are satisfied, then there exists a function $F(z)$ and a polynomial $P(z)$ in $1/z$ such that $F(z)$ is analytic interior to C, continuous in the corresponding closed region, and such that

$$F(z) + P(z) = f(z), \quad z \text{ on } C.$$

The polynomial $P(z)$ is uniquely determined if we require that it shall vanish at infinity; otherwise is uniquely determined only to within an additive constant.

An alternate statement for Theorem XII is

THEOREM XIIa. *If the function $f(z)$ is continuous on the analytic Jordan curve C, and if we have*

$$\int_C f(z)\omega(z)dz = 0, \tag{27}$$

for every function $\omega(z)$ analytic in the closed region interior to C, then there exists a function $F(z)$ analytic interior to C, continuous in the closed region which consists of C and its interior, and which coincides with $f(z)$ on C.*

The equivalence of (26) and (27) is easy to show. If (27) holds, (26) is surely satisfied. If (26) holds, then an arbitrary function $\omega(z)$ of the kind considered in Theorem XIIa can be uniformly expanded, in the closed region consisting of C and its interior, in a series of polynomials:

$$\omega(z) = \pi_0(z) + \pi_1(z) + \pi_2(z) + \cdots.$$

Term-by-term multiplication of this series by $f(z)$ and term-by-term integration yield, by means of (26), equation (27). Thus (27) is both a necessary and a sufficient condition for the existence of $F(z)$.

There is a similar statement for functions representing the boundary values of a function analytic at infinity:

* We have here an equivalent condition if $\omega(z)$ is required to be analytic merely interior to C and continuous in the corresponding closed region.

THEOREM XIII. *Let the function $f(z)$ be continuous on the analytic Jordan curve C, in whose interior the origin lies; then the two equivalent conditions*

$$\text{(A)} \qquad \int_C f(z)z^k dz = 0 \qquad (k = -1, -2, -3, \cdots);$$

$$\text{(B)} \qquad \int_C f(z)\omega(z)dz = 0,$$

for every function $\omega(z)$ analytic exterior to C (also at infinity), continuous in the corresponding closed region, and zero at infinity—these two conditions are each necessary and sufficient that there should exist a function $F(z)$ zero at infinity, analytic exterior to C (including the point at infinity), continuous in the corresponding closed region, and equal to $f(z)$ on C.

The proof of this theorem is easy and will be omitted.

If in condition (A) we omit $k=-1$, and in (B) require that $\omega(z)$ should have a double root at infinity, those two conditions remain equivalent. The conditions are then necessary and sufficient for the existence of $F(z)$, analytic exterior to C (including the point at infinity), continuous in the corresponding closed region, and equal to $f(z)$ on C. We cannot say, however, that $F(z)$ vanishes at infinity.

It will be noticed that if the continuous function $f(z)$ satisfies (26) as well as (A) of Theorem XIII, the two functions defined interior and exterior to C respectively are analytic in the neighborhood of C, hence analytic also on C. The function analytic exterior to C vanishes at infinity, so $f(z)$ is identically zero.

12. **Extension to multiply-connected regions.** In extending Theorems XII and XIII to the case of regions bounded by several contours, we shall mention merely the analogue of condition (A) although the analogue of condition (B) is easily included.

THEOREM XIV. *If the analytic Jordan curve C' lies interior to the analytic Jordan curve C, if the origin lies interior to C', and if the functions $f_1(z)$ and $f_2(z)$ continuous on C and C' respectively satisfy the conditions*

$$(28) \qquad \int_C f_1(z)z^k dz = \int_{C'} f_2(z)z^k dz \qquad (k = \cdots -2, -1, 0, 1, 2, \cdots),$$

then there exists a function $F(z)$ analytic in the annular region bounded by C and C', continuous in the corresponding closed region, and which on C and C' coincides with $f_1(z)$ and $f_2(z)$ respectively.

It is sufficient to establish Theorem XIV in the case that C is a circle. For if the theorem is true in that case, we shall prove it to be true in the general

case. Let $z=\psi(x)$, $x=\phi(z)$ denote as usual the functions which map the interior of C onto the interior of the unit circle γ in the x-plane. Since $\phi(0)=0$, the curve C' corresponds to an analytic Jordan curve γ' in whose interior the origin $x=0$ lies.

Conditions (28) lead to the equations

$$\int_C f_1(z)[\phi(z)]^k\phi'(z)dz = \int_{C'} f_2(z)[\phi(z)]^k\phi'(z)dz \tag{29}$$
$$(k = \cdots -2, -1,0,1,2, \cdots).$$

For the function $[\phi(z)]^k\ \phi'(z)$ is analytic in the closed region bounded by C and C', hence in that closed region can be uniformly expanded in a series of polynomials in z and $1/z$.* This expansion can be integrated term by term on C or C', after multiplication through by $f_1(z)$ or $f_2(z)$. Computation of the two members of (29) by the use of this series makes their equality evident in the light of (28).

Equations (29) are precisely the equations

$$\int_\gamma f_1[\psi(x)]x^k dx = \int_{\gamma'} f_2[\psi(x)]x^k dx \qquad (k = \cdots -2, -1,0,1,2, \cdots),$$

sufficient for the existence of a function $f(x)$ analytic in the annular region bounded by γ and γ', continuous in the corresponding closed region, and equal to $f_1[\psi(x)]$ and $f_2[\psi(x)]$ on γ and γ'; this is sufficient for the existence of the function $F(z)$ of the theorem.

It remains, then, to prove Theorem XIV when C is a circle. Consider the formal Taylor development of the function $f_1(z)$:

$$f_1(z) \sim a_0 + a_1 z + a_2 z^2 + \cdots, \quad a_k = \frac{1}{2\pi i}\int_C \frac{f_1(z)}{z^{k+1}}dz.$$

This series converges interior to C, defining a function $F_1(z)$ analytic interior to C, and the series converges uniformly on any curve Γ interior to C. Thus if Γ is an arbitrary rectifiable Jordan curve interior to C and includes in its interior the origin, we have

$$a_{k-1} = \int_C f_1(z)z^{-k}dz = \int_\Gamma F_1(z)z^{-k}dz \quad (k = 1,2,3, \cdots).$$

Hence the function $f_2(z) = F_1(z)$ is continuous on C' and satisfies the conditions

* This is very easy to prove by writing the function involved as the sum of two functions, given by Cauchy's integral taken over C and C' respectively. The one function is analytic on and interior to C, the other on and exterior to C'.

$$\int_{C'} [f_2(z) - F_1(z)]z^{-k}dz = \int_{C'} f_2(z)z^{-k}dz - \int_C f_1(z)z^{-k}dz = 0$$

$$(k=1, 2, 3, \cdots).$$

By Theorem XIII there exists a function $F_2(z)$ which is analytic exterior to C', continuous in the corresponding closed region, and which on C' coincides with $f_2(z) - F_1(z)$. The function $F_2(z) - f_1(z)$ is continuous on C. From the relations

$$\int_C F_2(z)z^k dz = \int_{C'} F_2(z)z^k dz = \int_{C'} f_2(z)z^k dz - \int_{C'} F_1(z)z^k dz$$

$$= \int_{C'} f_2(z)z^k dz = \int_C f_1(z)z^k dz \qquad (k = 0,1,2, \cdots),$$

we deduce by Theorem XII the existence of a function $\Phi(z)$ analytic interior to C, continuous in the corresponding closed region, and coinciding on C with $F_2(z) - f_1(z)$:

$$\Phi(z) = F_2(z) - f_1(z), \qquad z \text{ on } C. \tag{30}$$

We have, however,

$$-\int_C \Phi(z)z^k dz = \int_C [F_2(z) - \Phi(z)]z^k dz = \int_C f_1(z)z^k dz$$

$$(k = -1, -2, -3, \cdots),$$

so that the two functions $-\Phi(z)$ and $F_1(z)$ have the same coefficients in their Taylor development about the origin and are therefore identical.

The function $F_2(z) - \Phi(z)$ is then analytic interior to the annular region bounded by C and C', continuous in the corresponding closed region, by (30) equals $f_1(z)$ on C, and by the definition of $F_2(z)$ equals $f_2(z)$ on C'.

Theorem XIV, whose proof is now complete, can be extended to regions of higher connectivity:

THEOREM XV. *Let R be the region bounded by an analytic Jordan curve C_0 and by non-intersecting analytic Jordan curves $C_1, C_2, \cdots, C_n$ lying interior to C_0. If the function $f(z)$ is continuous on C, the complete boundary of R, then a necessary and sufficient condition that there exist a function $F(z)$ analytic in R, continuous in the corresponding closed region, and equal to $f(z)$ on C, is*

$$\int_C f(z)z^k dz = 0 \qquad (k = 0,1,2,\cdots),$$

(31)

$$\int_C f(z)(z - z_i)^k dz = 0 \qquad (i = 1,2,\cdots, n;\ k = 1,2,3,\cdots),$$

where z_i is an arbitrary fixed point interior to C_i. The integrals in (31) *are to be taken over the complete boundary of R, in the positive sense on that boundary.*

The proof of Theorem XV is similar to the proof of Theorem XIV and is omitted. Theorem XV remains true if the Jordan curves C_i are no longer required to be analytic, provided they are regular in the sense of Osgood,* and provided the function $f(z)$ satisfies a Lipschitz condition on C. The proof of this new theorem is likewise fairly simple, as an application of the theorem of Plemelj used in I, p. 167. It will be noticed too that in the proofs of the theorems given we need not require that the curves used be analytic; it is sufficient if the derivative $\phi'(z)$ of the mapping function used in each case is continuous in the closed region which we map.

The existence of other theorems which lie not far away is obvious; we give a single example related to Theorem XIV:

THEOREM XVI. *Let the analytic Jordan curve C′ lie interior to the analytic Jordan curve C, let the origin lie interior to C′, and let the functions $f_1(z)$ and $f_2(z)$ be defined and continuous on C and C′ respectively. Then a necessary and sufficient condition for the existence of a function $F(z)$ analytic interior to C, continuous in the corresponding closed region, and coinciding on C and C′ with $f_1(z)$ and $f_2(z)$ respectively, is*

$$\int_C f_1(z)z^k dz = \int_{C'} f_2(z)z^k dz = 0 \qquad (k = 0,1,2,\cdots),$$

$$\int_C f_1(z)z^k dz = \int_{C'} f_2(z)z^k dz \qquad (k = -1,-2,-3,\cdots).$$

Theorems XII–XVI have obvious application to expansions in terms of polynomials, particularly in connection with such theorems as IX, which do not demand analyticity for the development of a given function.

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* *Funktionentheorie*, Leipzig, 1912, p. 51.

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PRIMITIVE GROUPS WHICH CONTAIN SUBSTITUTIONS OF PRIME ORDER p AND OF DEGREE $6p$ OR $7p$*

BY

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1. In a memoir on primitive groups in the first volume of the Bulletin of the Mathematical Society of France† Jordan announced the following theorem:

Let q be a positive integer < 6, p any prime $>q$. The degree of a primitive group G that contains a substitution of order p on q cycles (without including the alternating group) cannot exceed $pq+q+1$.

He gave proofs for the cases $q=1$, and $q=2$, but no proofs for greater values of q. In the same memoir, he also found a limit for the degree of G when q is not restricted to numbers <6. His results may be stated as follows:

Let q be any positive integer, p any prime $>2q\log_2 q+q+1$. The degree of a primitive group G that contains a substitution of order p on q cycles (without including the alternating group) cannot exceed $qp+2q\log_2 2q$.

Manning has studied this problem further. He not only gave proofs for the cases $q=2, 3, 4, 5$, finding a somewhat closer limit than the one announced by Jordan, but also found a much lower limit for the degree of G in the general case. Using the theory developed in the proof of his general theorem, he investigated the case $q=6$. A brief statement of these theorems follows:

Let q be any integer greater than unity and <5, p any prime $>q+1$. Then the degree of a primitive group of class >3 which contains a substitution of order p and of degree qp cannot exceed $qp+q$. When $p=q+1$, the degree cannot exceed $qp+q+1$.

If a primitive group of class >3 contains a substitution of prime order p on 5 cycles, $p>5$, its degree cannot exceed $5p+6$. Moreover, if a primitive group of degree $5p+6$ exists, it is doubly transitive.

The degree of a primitive group G of class >3 which contains a substitution of prime order p and of degree $qp(p>2q-3, q>1)$, does not exceed $qp+4q-4$. Moreover p^2 does not divide the order of G.

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† C. Jordan, Bulletin de la Société Mathématique de France, vol. 1 (1873), pp. 175–221.

The degree of a primitive group of class >3 *which contains a substitution of prime order* p *and of degree* $6p(p>6)$ *does not exceed* $6p+10$.

He published these results in a series of 4 papers, entitled *On the order of primitive groups*.* This title draws attention to the fact that the problem of finding a limit for the degree of these primitive groups is equivalent to the problem of finding a limit for the order of a primitive group in terms of its degree. The fact that the order of these groups is limited by their degree is discussed by Manning in his second paper.

2. In the present paper, the case $q=7$ will be investigated and the limit for the degree of G for the case $q=6$ will be lowered. The following theorems will be proved:

The degree of a primitive group G *of class* >3 *which contains a substitution of prime order* $p(p>7)$ *and of degree* $6p$ *cannot exceed* $6p+6$. *If* $p=7$, *the true limit for the degree of* G *is* $6p+7$. *Moreover, if* G *of degree* $6p+7$ *exists, it is doubly transitive.*

The degree of a primitive group G *of class* >3 *which contains a substitution of prime order* p *and of degree* $7p$, $p>7$, *does not exceed* $7p+8$. *Moreover, if* G *of degree* $7p+8$ *exists, it is doubly transitive.*

Although the first theorem is not proved until §17, we shall use it in the proof of the second theorem, for the former depends in no way upon the latter. The proof of the latter theorem is based on the general method developed by Manning in his third paper *On the order of primitive groups*, of which especially §§9–20 should be carefully read before reading the following proof. All definitions and the fundamental theory will be found in these sections. The assumption (III, §12) that the degree of H_{r+1} exceeds $qp+q$ should be noted, for with the exception of §15 this hypothesis is held throughout the present paper. We now proceed to the proof of this theorem.

3. If H_{r+1} is imprimitive, H_{r+s} (III, §18) has systems of imprimitivity of 7 letters only, for the number of letters in a system must divide 7 (III, §17). Moreover, if H_{r+1} is of degree $>qp+q$, the systems of imprimitivity of H_{r+s} are permuted according to a primitive group which is not triply transitive (III, §18), thus according to a primitive group of degree $p+1$ at most. Then the degree of H_{r+s} does not exceed $7p+7$. Now since a generator introduces 7 letters or none, $s=1$. We shall now consider H_{r+1} of degree $7p+7$. Clearly the order of H_{r+1} is not divisible by p^2 when $p>7$. Then J_1

* W. A. Manning, these Transactions, vol. 10 (1909), p. 247; vol. 16 (1915), p. 139; vol. 19 (1918), p. 127; vol. 20 (1919), p. 66. These 4 papers will be referred to by the Roman numerals I. II, III, IV, respectively.

is transitive of degree 7. Now H_{r+1} may be contained in a doubly transitive group of degree $7p+8$, for if H_{r+2} exists, J_2 is multiply transitive of degree 8. Since the J group is always a transitive representation of a subgroup or quotient group of the direct product of a cyclic group of degree a divisor of $p-1$ and the symmetric group of degree 7 (see §5), J_2 must be the simple group of order 168, for it is the only primitive group of degree 8 which does not occur for the first time on 8 letters. A triply transitive group of degree $7p+9$ does not exist, for then J_3 is of degree 9. However, J_2 is not contained in any non-alternating primitive group of higher degree, and J is not alternating when its degree exceeds 7 (III, §20).

4. Let H_{r+1} be a primitive group. Its subgroup F is intransitive. If $p>2q-3$, no constituent of F is alternating, nor does an imprimitive constituent permute its systems according to an alternating group (III, §35).*

We shall now assume $p>2q-3$. The case of $p=2q-3$ will be taken up in §16. Then the degree of F does not exceed $7p+14$. The order of F is not divisible by p^2 (III, §27), nor is the order of the subgroup L of H_{r+1} that leaves one letter fixed.

5. The group I_1 that occurs in H_{r+1} has no substitutions on the letters of J_1 only, for then G would be of class $\leqq 2q-4$ (III, §21) and therefore of degree $\leqq 25$.†

We shall need the following theorems (I, Theorems 5 and 6, p. 251) in discussing the J groups.

Let P be a cyclic group of prime order p and of degree $qp(q<p)$. The largest group G on the same letters that transforms P into itself and that contains no substitution of order p with $<q$ cycles is of order $p(p-1)(q!)$.

The quotient group G/P is the direct product of a cyclic group of order $p-1$ and a group isomorphic to the symmetric group of degree q.

Then the constituent of I_1 on the letters of A_1 is the group of order $p(p-1)(q!)$ or a subgroup of it. Thus J_1 is a transitive representation of the direct product of a cyclic group of order d (d a divisor of $p-1$) and a group

* An accurate statement of this theorem follows: If $p>2q-3$, the degree of H_{r+1} does not exceed $qp+q$, when H_{ij} has a transitive constituent which is alternating or which permutes systems of imprimitivity according to an alternating group. If $p=2q-3$, there is one exception to this limit of the degree of H_{r+1}. It occurs when E_1 (III, § 30) has a transitive constituent simply isomorphic to the alternating constituent of degree p. Then E_1 has three constituents of degrees $(p-1)p/2, p, p$. The third constituent cannot be of degree $p+k$, for such a constituent is of too small a degree to be simply isomorphic to the alternating constituent of degree p and (III, §33) E_1 cannot have a primitive constituent multiply isomorphic to the alternating constituent of degree p. Then H_{r+1} is of degree $\leqq qp+2q$.

† W. A. Manning, American Journal of Mathematics, vol. 28 (1906), p. 226.

K which occurs as a quotient group among the groups of degree $\leqq 7$. Let the group K of order kk' be multiplied into a cyclic group of order d. The direct product of order $kk'd$ can be represented as a transitive group on dk letters if and only if the group K of order kk' has a subgroup K' of order k' which contains no invariant subgroup of K. Call the subgroup of J_1 that leaves one letter fixed J_1'. It should be noted that J_1' is not the identity if H_{r+1} is of degree $>qp+q$. If the subgroup K' is invariant in a group of order mk', J_1', isomorphic to K', is invariant in a group of order mk' and therefore fixes exactly m letters.

6. The following theorems on simply transitive primitive groups by Manning will be used repeatedly:

THEOREM 1. *Let G_1, the subgroup that fixes one letter of a simply transitive primitive group G of degree n and order g, have a multiply transitive constituent of degree m. If G_1 has no transitive constituent whose degree $(>m)$ is a divisor of $m(m-1)$, all the transitive constituents of G_1 are simply isomorphic multiply transitive groups of degree m and order g/n.**

THEOREM 2. *If only one transitive constituent of G_1 is an imprimitive group (of order f), G_1 is of order f.*

THEOREM 3. *If G_1 has an intransitive constituent of order f, and if all the transitive constituents on the remaining letters of G_1 are primitive groups, G_1 is of order f.*†

THEOREM 4. *Let G_1 have a primitive constituent M of degree m, in which the subgroup M_1 that fixes one letter is primitive. Let M be paired with itself in G_1 and let the order of M be $<g/n$. Then G_1 contains an imprimitive constituent in which there is an invariant intransitive subgroup with m transitive constituents of $m-1$ letters each, permuted according to the permutations of the primitive group M.*‡

7. We wish to see how far the reasoning used in the proof of Theorem 1 may be applied to the subgroup F of H_{r+1} when H_{r+1} is a primitive group of degree $>qp+q$ $(q>5,\ p>2q-3)$. Let $L(x)$ $(=L$, III, §21) be the subgroup of H_{r+1} that fixes the letter x. Then $L(x)$ has an invariant subgroup $F(x)=F$ generated by all of its substitutions of order p. Let $F(x)$ have a transitive constituent of degree $p+2$ on the letters $a_1, a_2, \cdots, a_{p+2}$. If the order of $F(x)$ is t, the order of $F(x)(a_1)$ is $t/(p+2)$ and the order of $F(x)(a_1)(a_2)$

* W. A. Manning, *Primitive Groups*, 1921, p. 83.

† W. A. Manning, Proceedings of the National Academy of Sciences, vol. 12 (1926), p. 755.

‡ W. A. Manning, these Transactions, vol. 29 (1927) pp. 815–825.

is $t/[(p+2)(p+1)]$. In $F(a_1)$, x belongs to a transitive constituent of degree $p+2$. Since $F(a_1)(x)$ has a transitive constituent on the letters $a_2, a_3, \cdots, a_{p+2}$, the transitive constituent to which x belongs in $F(a_1)$ contains either $p+1$ a's or none. Suppose that $F(a_1)$ has a transitive constituent on the letters $x, a_2, a_3, \cdots, a_{p+2}$. Then the group $\{F(a_1), F(x)\}$ has a transitive constituent of degree $p+3$ on the letters $x, a_1, a_2, \cdots, a_{p+2}$. Now a group of degree $p+3$ that contains a substitution of degree and order p is alternating. The subgroup of $\{F(a_1), F(x)\}$ that fixes the letter x contains an invariant subgroup, $F(x)$, generated by all of its substitutions of order p and has an alternating constituent on the letters $a_1, a_2, \cdots, a_{p+2}$. Since an alternating group of degree >4 is simple, $F(x)$ has an alternating constituent on the letters $a_1, a_2, \cdots, a_{p+2}$. However, if $p>2q-3$, F has no alternating constituents. If $F(a_1)$ has a transitive constituent on the letters $a_2, a_3, \cdots, a_{p+2}$, $F(x)$ also has a transitive constituent of degree $p+1$. But it can be shown that F cannot have constituents of degree $p+2$ and $p+1$ at the same time. The constituent groups of F are positive groups. In order that the constituent of degree $p+2$ contain no negative substitutions, the substitution from its I group that is associated* with the substitution from its J group must be negative. This negative substitution is from the metacyclic group and consequently it is negative in the I group of the constituent of degree $p+1$. Substitutions from the metacyclic group of I_1 have cycles on letters of each cycle of A_1. Then the letters $a_2, a_3, \cdots, a_{p+2}$ belong in $F(a_1)$ to a transitive constituent of degree $\mu \geqq p+2$. Thus the order of $F(a_1)(a_2)$ is t/μ, and if x belongs to a transitive constituent of degree δ in $F(a_1)(a_2)$, the order of $F(a_1)(a_2)(x)$ is $t/(\mu\delta)$. Then $t/[(p+2)(p+1)]=t/(\mu\delta)$. If $F(x)$ contains no constituent whose degree $(>p+2)$ divides $(p+2)(p+1)$, $\mu=p+2$.

The next difficulty in applying the proof of Theorem 1 arises when we consider the constituents of degree $p+2$ which contain $p+1$ a's in the groups $F(a_1), F(a_2), \cdots, F(a_{p+2})$. However, if $c_1=c_2$, the group $\{F(a_1), F(a_2)\}$ contains a transitive constituent on the letters $c_1, a_1, a_2, \cdots, a_{p+2}$. Thus as above, $F(a_1)$ has an alternating constituent on the letters $c_1, a_2, a_3, \cdots, a_{p+2}$.

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* An intransitive group may be regarded as formed from its transitive constituents by establishing an isomorphism between one transitive constituent and the constituent (transitive or intransitive) on the remaining letters, and then multiplying corresponding substitutions. Thus any substitution of an intransitive group is the product of substitutions from all transitive constituents (taking the identity into account). These substitutions from different transitive constituents which occur as factors in a given substitution are said to be *associated*. For example, in the intransitive octic group written out in §13, the substitution (47) is said to be associated with the substitution (58) (69).

We may now follow the proof of Theorem 1 until the statement "if B and C coincide." If B and C coincide, the group $\{F(a_1), F(x)\}$ has a transitive constituent of degree $2p+5$. Such a constituent is alternating. Now the proof (III, §28) that no alternating constituent of H_{ij} involves letters of more than one cycle of A_1 without causing the presence of a substitution of order p and of degree $<qp$ in G applies to any intransitive group generated by substitutions of order p and of degree qp, thus also to the group $\{F(a_1), F(x)\}$. Since transitive groups generated by substitutions of order p and of degrees $3p+7$, $4p+9$, $5p+11$, $6p+13$ are alternating, we see that if $F(x)$ has one transitive constituent of degree $p+2$ and no transitive constituent whose degree $(>p+2)$ divides $(p+2)(p+1)$, it has at least six constituents of degree $p+2$.

The results of the above discussion may be summarized in

THEOREM 5. *Let H_{r+1} be a primitive group of degree $>qp+q$ $(p>2q-3$, $q>5)$. If F has a transitive constituent of degree $p+2$ and no transitive constituent whose degree $(>p+2)$ divides $(p+2)(p+1)$, it has at least six transitive constituents of degree $p+2$.*

The following theorem will also be useful.

THEOREM 6. *Let G be a simply transitive primitive group. If the subgroup that fixes one letter of G has a transitive constituent of degree m, it must also have another transitive constituent whose degree divides mk_i, where k_i $(\geqq 1)$ is the degree of a transitive constituent of the subgroup that fixes one letter of the constituent of degree m.*

Let $G(x)$ be the subgroup of G that fixes the letter x. Let $G(x)$ have a transitive constituent of degree m on the letters $a_1, a_2, \cdots, a_m$. Note that the theorem is proved if $G(x)$ has a second transitive constituent of degree m. It may thus be assumed that $G(x)$ has only one transitive constituent of degree m. Now let $G(x)(a_1)$ have transitive constituents of degrees k_1, $k_2, \cdots, k_v$ on the letters $a_2, a_3, \cdots, a_m$. The order of $G(x)$ is g/n, if n is the degree of G. Then the order of $G(x)(a_1)$ is $g/(nm)$, and the order of $G(x)(a_1)(a_2)$ is $g/(nmk_1)$, $g/(nmk_2)$, $\cdots$, or $g(nmk_v)$, according as a_2 belongs to a transitive constituent of degree $k_1, k_2, \cdots$, or k_v in $G(x)(a_1)$. Now x belongs to a transitive constituent of degree m in $G(a_1)$. We know that at least one of the transitive constituents of degree k_i, $i=1, 2, \cdots, v$, on the letters $a_2, a_3, \cdots, a_m$ in $G(x)(a_1)$ belongs in $G(a_1)$ to a transitive constituent of degree $r>k_i$, which does not include the letter x.* Then the order of

* W. A. Manning, these Transactions, vol. 29 (1927), p. 815.

$G(a_1)(a_2)$ is $g/(nr)$, if a_2 is the letter that occurs in the transitive constituent of degree k_i in $G(x)(a_1)$, and if x belongs to a transitive constituent of degree s in $G(a_1)(a_2)$, the order of $G(a_1)(a_2)(x)$ is $g/(nrs)$. Then at least one of the following equations is true:

$$r = mk_i/s \qquad (i = 1, 2, \cdots, v).$$

If $k=1$, a condition that may arise when the constituent of degree m is imprimitive, we conclude that r is a divisor of m.

8. We are now prepared to study the subgroup F of a primitive H_{r+1}. Let F be of degree $7p+14$ (§4). Since F includes H_3, it has at most 5 constituents. The partitions of the degree of F are the following:

$$\begin{array}{lllll}
6p+12, & p+2 & & & \\
5p+10, & 2p+4 & & & \\
4p+8, & 3p+6 & & & \\
5p+10, & p+2, & p+2 & & \\
4p+8, & 2p+4, & p+2 & & \\
3p+6, & 3p+6, & p+2 & & \\
3p+6, & 2p+4, & 2p+4 & & \\
4p+8, & p+2, & p+2, & p+2 & \\
3p+6, & 2p+4, & p+2, & p+2 & \\
2p+4, & 2p+4, & 2p+4, & p+2 & \\
3p+6, & p+2, & p+2, & p+2, & p+2 \\
2p+4, & 2p+4, & p+2, & p+2, & p+2.
\end{array}$$

The following partitions are impossible: $6p+12$, $p+2$; $4p+8$, $2p+4$, $p+2$; $3p+6$, $3p+6$, $p+2$; $2p+4$, $2p+4$, $2p+4$, $p+2$, for in each case L has a transitive constituent of degree $p+2$, paired with itself, whose order is less than that of L. By Theorems 2 and 3, the imprimitive constituents of L determine the order of L. The multiple isomorphism between the constituent of degree $p+2$ and the constituent whose order determines that of L follows from the multiple isomorphism between the J groups of these constituents. Then all the conditions of Theorem 4 are satisfied and consequently L should have a transitive constituent of degree $(p+2)(p+1)$.

Now J_1 is a transitive group of degree 15. Then $dk=15$, and $d=1$, 3, or 5. When $d=3$ or 5, J_1' fixes 3 and 5 letters respectively. However, the partitions of the degree of F show that J_1' fixes one letter only and therefore $d=1$ and $k=15$. Now the partitions also show that k' is even. No orders will be listed when there are no groups or quotient groups of these orders on <8 letters. Then the possible values of $15k'$ are 60, 120, 240, 360, 720, 2520,

5040. The order 5040 is impossible, for the symmetric group of degree 7 contains no subgroup of order 336.

If $15k'=60$, J_1' is of order 4. Since the J group of a constituent of degree $2p+4$ is octic (IV, §3), the partitions of the degree of F exclude this representation.

Let $15k'=120$. J_1' is of order 8. Since the least order of the J group of a constituent of degree $4p+8$ is 16 (IV, §3), we need consider for this representation only the following partition of the degree of F: $2p+4$, $2p+4$, $p+2$, $p+2$. From the groups of order 120 on <8 letters, only one distinct representation is obtained. This representation is given by the symmetric group of degree 5 with respect to its octic subgroup. J_1' has three transitive constituents of degrees 2, 4, and 8. It is generated by

$$\{23\cdot 4567\cdot 8yux\cdot 9zvo,\ \ 23\cdot 46\cdot 8z\cdot ou\cdot xv\cdot 9y\}.$$

Then in L this partition of the degree of F becomes $4p+8$, $2p+4$, $p+2$. Thus the constituent of degree $p+2$ in L satisfies all the conditions of Theorem 4, but L has no transitive constituent of degree $(p+2)(p+1)$.

If $15k'=240$, J_1' is of order 16. However the only group of order 240 on <8 letters contains no non-invariant subgroup of order 16.

If $15k'=360$, J_1' is of order 24. There is only one representation of the alternating group of degree 6 on 15 letters and J_1' has two transitive constituents one of degree 6 and one of degree 8. The group J_1' is generated by

$$\{25\cdot 34\cdot 68\cdot 79\cdot xy\cdot zu,\ \ 28\cdot 36\cdot 45\cdot 79\cdot ox\cdot vu,\ \ 29\cdot 37\cdot 45\cdot 68\cdot oz\cdot yv\}.$$

Now consider the possible partitions of the degree of F for this J_1'. A constituent of degree $3p+6$ is impossible, for its J group is of order 18, 36, or 72 (IV, §3). A constituent of degree $4p+8$ is likewise impossible, for its J group is of order 16, 32, 128 or greater. Then there is only the partition $2p+4$, $2p+4$, $p+2$, $p+2$, $p+2$ to be considered. It calls for an invariant intransitive subgroup of degree and order 8 in J_1'. However all the subgroups of order 8 are conjugate in J_1'.

If $15k'=720$, J_1' is of order 48. There is one representation of the symmetric group of degree 6 with respect to its subgroup $\{abc, ad, ef\}$. J_1' has two constituents of degrees 8 and 6, respectively, in a two-to-one isomorphism. It is generated by

$$\{246\cdot 573\cdot ozv\cdot xuy,\ \ 28\cdot 39\cdot xz\cdot yv,\ \ 23\cdot 45\cdot 67\cdot 89\}.$$

Now the J group of a constituent of degree $3p+6$ in F is incompatible with a J_1' of order 48. Then the only possible partitions are $4p+8$, $p+2$, $p+2$,

$p+2$, and $2p+4$, $2p+4$, $p+2$, $p+2$, $p+2$. Neither partition, however, allows a two-to-one isomorphism between the constituents of J_1'.

If $15k'=2520$, there is one representation of the alternating group of degree 7 with respect to its subgroup of order 168. Now this subgroup contains substitutions of order 7 and consequently J_1' contains substitutions of the same order. Then J_1' has two constituents of degree 7 or it is transitive of degree 14. The partitions of the degree of F show that either case is impossible. Hence H_{r+1} is not of degree $7p+15$.

9. Let F be of degree $7p+13$. The partitions of the degree of F are the following:

$6p+12$,	$p+1$				
$5p+10$,	$p+2$,	$p+1$			
$4p+8$,	$2p+4$,	$p+1$			
$3p+6$,	$3p+6$,	$p+1$			
$4p+8$,	$p+2$,	$p+2$,	$p+1$		
$3p+6$,	$2p+4$,	$p+2$,	$p+1$		
$2p+4$,	$2p+4$,	$2p+4$,	$p+1$		
$3p+6$,	$p+2$,	$p+2$,	$p+2$,	$p+1$	
$2p+4$,	$2p+4$,	$p+2$,	$p+2$,	$p+1$	
$2p+4$,	$p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+1$.

All these partitions have one and only one constituent of degree $p+1$. Then by Theorem 1, L should have a transitive constituent whose degree divides $(p+1)p$. Thus H_{r+1} is not of degree $7p+14$.

10. H_{r+1} cannot be of degree $7p+13$. If $p>13$, J_1 is transitive of degree 13, but no subgroup of the direct product of a cyclic group of order a divisor of $p-1$ and the symmetric group of degree 7 can be written as a group of degree 13. Then $p=13$. However, a primitive group of degree $qp+p$ does not exist unless $p<2q-2$ (III, §22).

Similarly if H_{r+1} is of degree $7p+11$, $p=11$, but by hypothesis $p>11$.

11. Let F be of degree $7p+11$. The partitions of the degree of F are the following:

$5p+10$,	$2p+1$		
$4p+8$,	$3p+3$		
*$5p+10$,	$p+1$,	p	
*$4p+8$,	$2p+2$,	$p+1$	
†$4p+8$,	$2p+1$,	$p+2$	
†$3p+6$,	$3p+3$,	$p+2$	
$3p+6$,	$2p+4$,	$2p+1$	
$3p+3$,	$2p+4$,	$2p+4$	
*$4p+8$,	$p+2$,	$p+1$,	p

$4p+8$,	$p+1$,	$p+1$,	$p+1$		
*$3p+6$,	$2p+4$,	$p+1$,	p		
*$3p+6$,	$2p+2$,	$p+2$,	$p+1$		
$3p+6$,	$2p+1$,	$p+2$,	$p+2$		
$3p+3$,	$2p+4$,	$p+2$,	$p+2$		
*$2p+4$,	$2p+4$,	$2p+2$,	$p+1$		
†$2p+4$,	$2p+4$,	$2p+1$,	$p+2$		
*$3p+6$,	$p+2$,	$p+2$,	$p+1$,	p	
†$3p+6$,	$p+2$,	$p+1$,	$p+1$,	$p+1$	
$3p+3$,	$p+2$,	$p+2$,	$p+2$,	$p+2$	
*$2p+4$,	$2p+4$,	$p+2$,	$p+1$,	p	
$2p+4$,	$2p+4$,	$p+1$,	$p+1$,	$p+1$	
*$2p+4$,	$2p+2$,	$p+2$,	$p+2$,	$p+1$	
$2p+4$,	$2p+1$,	$p+2$,	$p+2$,	$p+2$	
*$2p+4$,	$p+2$,	$p+2$,	$p+2$,	$p+1$,	p
‡$2p+4$,	$p+2$,	$p+2$,	$p+1$,	$p+1$,	$p+1$
*$2p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+1$
**$2p+1$,	$p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+2$.

In this and the remaining sections all the partitions of the degree of F which are impossible by Theorem 1 are prefixed by the asterisk *. Likewise all partitions which are impossible by Theorem 4 are prefixed by the dagger † and those which contradict Theorem 5 by the two asterisks **.

In the present case, among the partitions which remain after eliminating those impossible by Theorems 1, 4, and 5, the partition $2p+4$, $p+2$, $p+2$, $p+1$, $p+1$, $p+1$ is also impossible, for F cannot have constituents of degrees $p+2$ and $p+1$ at the same time without containing a negative substitution (see §7). In the remaining cases, partitions of the degree of F which are impossible for this reason will be prefixed by the double dagger ‡.

Now consider the possible J groups. The 10 partitions of the degree of F that remain to be considered show that the order of J_1' is even. Then if $d=1$, $k=12$, and $12k'=24$, 48, 72, 120, 144, 168, 240, 360, 720, 2520, 5040. The orders 360, 2520, 5040 are impossible, for there are no groups of orders 30, 210, or 420 on <8 letters. Since there are no partitions which allow J_1' to be of order 2, $12k'\neq 24$. Neither are there any partitions which allow J_1' to be of order 4, for the J group of a constituent of degree $2p+4$ is octic.

If $12k'=72$, the only possible partition is $3p+3$, $p+2$, $p+2$, $p+2$, $p+2$, for the J group of a constituent of degree $3p+6$ is at least of order 18. This partition brings a substitution of degree and order 3 into J_1'. The group J_1 is imprimitive, for a primitive J_1 would be alternating, but if J_1 is of degree

$>q$, it is not alternating (III, §20). Then J_1 has systems of imprimitivity of 3, 4, or 6 letters and its order is 324, 648, or greater.

If $12k'=120$, no partition of the degree of F is possible, for the least order of the J group of a constituent of degree $5p+10$ is 50 (IV, §3). If $12k'=144$ or 720, the only possible partition is again $3p+3$, $p+2$, $p+2$, $p+2$, $p+2$, but as has been seen this partition is incompatible with the order of J_1'. When $12k'=168$ or 240, no partition of the degree of F allows J_1' to be of order 14 or 20.

If $d=2$, $k=6$, and $6k'=12$, 24, 36, 48, 60, 72, 120, 360, 720. As we have seen, no partition of the degree of F allows J_1' to be of orders 2, 4, 6, 10, 12, 20, or 60.

Now when $d=2$, J_1' is invariant in a subgroup of twice its order and therefore fixes two letters of J_1. Then when $d>1$, J_1 may be constructed by first writing down the transitive representation of K on k letters with respect to its subgroup of order k' and then making it simply isomorphic to itself in d different sets of letters and in such a way that the subgroup of order d permutes these d transitive constituents cyclically and is commutative with each substitution of K. With the above in mind consider the case when J_1' is of order 8. There are only two possible partitions of the degree of F: $2p+4$, $2p+4$, $p+1$, $p+1$, $p+1$, and $2p+4$, $2p+1$, $p+2$, $p+2$, $p+2$. In the second partition since the constituent of degree $2p+4$ gives J_1' a constituent of degree 4, J_1' must have two constituents of degree 4. Thus J_1' is of degree 8 and this partition is then incompatible with such a J_1'. The first partition is also impossible because all the constituents of degree $p+1$ cannot unite in L if J_1' fixes two letters. Then L has a multiply transitive constituent of degree $p+1$ and is impossible by Theorem 1.

If J_1' is of order 120, it has two constituents of degree 5. However no partition of the degree of F is possible.

If $d=3$, $k=4$, and J_1' is invariant in a group of three times its order and therefore it fixes three letters. The only possible partitions of the degree of F, $4p+8$, $p+1$, $p+1$, $p+1$, and $2p+4$, $2p+4$, $p+1$, $p+1$, $p+1$, have a multiply transitive constituent of degree $p+1$ in L, for the constituents of degree $p+1$ cannot unite in L if J_1' fixes three letters. These partitions are then impossible by Theorem 1. If $d=4$, and $k=3$, the only possible partitions are the above and again they are impossible. Since no partition of the degree of F fixes so many as 6 letters, $d\neq 6$.

12. Let F be of degree $7p+9$. The partitions of the degree of F are the following:

$$5p+5,\quad 2p+4$$
$$4p+8,\quad 3p+1$$

$4p+3$,	$3p+6$			
*$5p+6$,	$p+2$,	$p+1$		
$5p+5$,	$p+2$,	$p+2$		
$4p+8$,	$2p+1$,	p		
$4p+8$,	$2p$,	$p+1$		
*$4p+4$,	$2p+4$,	$p+1$		
$3p+6$,	$3p+3$,	p		
*$3p+6$,	$3p+2$,	$p+1$		
†$3p+6$,	$3p+1$,	$p+2$		
††$3p+6$,	$2p+2$,	$2p+1$		
††$3p+3$,	$2p+4$,	$2p+2$		
$3p+1$,	$2p+4$,	$2p+4$		
$4p+8$,	$p+1$,	p,	p	
*$4p+4$,	$p+2$,	$p+2$,	$p+1$	
**$4p+3$,	$p+2$,	$p+2$,	$p+2$	
*$3p+6$,	$2p+2$,	$p+1$,	p	
†$3p+6$,	$2p+1$,	$p+2$,	p	
††$3p+6$,	$2p+1$,	$p+1$,	$p+1$	
†$3p+6$,	$2p$,	$p+2$,	$p+1$	
†$3p+3$,	$2p+4$,	$p+2$,	p	
††$3p+3$,	$2p+4$,	$p+1$,	$p+1$	
††$3p+3$,	$2p+2$,	$p+2$,	$p+2$	
*$3p+2$,	$2p+4$,	$p+2$,	$p+1$	
$3p+1$,	$2p+4$,	$p+2$,	$p+2$	
$2p+4$,	$2p+4$,	$2p+1$,	p	
$2p+4$,	$2p+4$,	$2p$,	$p+1$	
*$2p+4$,	$2p+2$,	$2p+2$,	$p+1$	
†$2p+4$,	$2p+2$,	$2p+1$,	$p+2$	
†$3p+6$,	$p+2$,	$p+1$,	p,	p
††$3p+6$,	$p+1$,	$p+1$,	$p+1$,	p
‡$3p+3$,	$p+2$,	$p+2$,	$p+2$,	p
‡$3p+3$,	$p+2$,	$p+2$,	$p+1$,	$p+1$
*$3p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+1$
**$3p+1$,	$p+2$,	$p+2$,	$p+2$,	$p+2$
$2p+4$,	$2p+4$,	$p+1$,	p,	p
*$2p+4$,	$2p+2$,	$p+2$,	$p+1$,	p
$2p+4$,	$2p+2$,	$p+1$,	$p+1$,	$p+1$
‡$2p+4$,	$2p+1$,	$p+2$,	$p+2$,	p
†$2p+4$,	$2p+1$,	$p+2$,	$p+1$,	$p+1$
‡$2p+4$,	$2p$,	$p+2$,	$p+2$,	$p+1$

*$2p+2$,	$2p+2$,	$p+2$,	$p+2$,	$p+1$	
**$2p+2$,	$2p+1$,	$p+2$,	$p+2$,	$p+2$	
‡$2p+4$,	$p+2$,	$p+2$,	$p+1$,	p,	p
†$2p+4$,	$p+2$,	$p+1$,	$p+1$,	$p+1$,	p
$2p+4$,	$p+1$,	$p+1$,	$p+1$,	$p+1$,	$p+1$
*$2p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+1$,	p
**$2p+2$,	$p+2$,	$p+2$,	$p+1$,	$p+1$,	$p+1$
**$2p+1$,	$p+2$,	$p+2$,	$p+2$,	$p+2$,	p
**$2p+1$,	$p+2$,	$p+2$,	$p+2$,	$p+1$,	$p+1$
**$2p$,	$p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+1$.

We now delete all those partitions of the degree of F which contradict Theorems 1, 4, and 5, and those which cause a constituent group of F to have a negative substitution. In the partitions preceded by the two daggers ††, J_1' has a substitution (IV, §3) of degree and order 3. If J_1 contains such a substitution it is imprimitive, for a primitive J_1 is alternating, and we know that J_1 is not alternating when its degree $>q$. The group J_1 has systems of imprimitivity of 5 letters only and its least possible order is 7200. Since a J_1 of this or greater order cannot be written on 7 or fewer letters, these partitions are impossible.

Now consider the possible J groups. If $d=1$, $k=10$, and $10k'=20$, 40, 60, 120, 240, 360, 720, 2520, 5040. Odd values of k' need not be considered, for the partitions of the degree of F show that the order of J_1' is even. Since there are no groups of order 252 or 504 on <8 letters, $10k'\neq 2520$, or 5040. The orders 20, 40, 60, and 120 are also impossible, for the partitions of the degree of F do not allow J_1' to be of order 2, 4, 6, or 12.

If $10k'=240$, the group $\{abcde, ab, fg\}$ may be represented on 10 letters by means of its symmetric subgroup of degree 4. This representation gives a J_1' with two constituents of degree 4 in a simple isomorphism. The only possible partitions of the degree of F are the following: $3p+1$, $2p+4$, $2p+4$; $2p+4$, $2p+4$, $2p+1$, p; $2p+4$, $2p+4$, $2p$, $p+1$; $2p+4$, $2p+4$, $p+1$, p, p. However these partitions are all impossible, for they all contain a constituent of degree $2p+4$ which demands that an octic group be invariant in the symmetric group of degree 4.

If $10k'=360$, the alternating group of degree 6 with respect to its subgroup $\{abc, def, aebd\cdot cf\}$ gives a doubly transitive J_1. Since there are no partitions which allow J_1 to be doubly transitive, $10k'\neq 360$. If $10k'=720$, the symmetric group of degree 6 with respect to its subgroup $\{ab, ac, de, df, ad\cdot be\cdot cf\}$ also gives a doubly transitive J_1.

If $d=2$, $k=5$, $5k'=10$, 20, 60, 120. We have seen that J_1' cannot be of order 2, 4, or 12. If $5k'=120$, we have a J_1' with two constituents of degree

4. Such a J_1' group has been seen to be impossible. (See this section, paragraph 4.)

13. Let F be of degree $7p+8$. The partitions of the degree of F are the following:

$^*6p+6$,	$p+2$		
$^*5p+6$,	$2p+2$		
$5p+4$,	$2p+4$		
$4p+8$,	$3p$		
$4p+2$,	$3p+6$		
$^*5p+6$,	$p+2$,	p	
$^*5p+6$,	$p+1$,	$p+1$	
$^*5p+5$,	$p+2$,	$p+1$	
$^{**}5p+4$,	$p+2$,	$p+2$	
$4p+8$,	$2p$,	p	
$4p+4$,	$2p+4$,	p	
$^*4p+4$,	$2p+2$,	$p+2$	
$^*4p+3$,	$2p+4$,	$p+1$	
$^\dagger 4p+2$,	$2p+4$,	$p+2$	
$3p+6$,	$3p+2$,	p	
$^*3p+6$,	$3p+1$,	$p+1$	
$^\dagger 3p+6$,	$3p$,	$p+2$	
$3p+3$,	$3p+3$,	$p+2$	
$3p+6$,	$2p+2$,	$2p$	
$3p+6$,	$2p+1$,	$2p+1$	
$3p+3$,	$2p+4$,	$2p+1$	
$3p+2$,	$2p+4$,	$2p+2$	
$3p$,	$2p+4$,	$2p+4$	
$4p+8$,	p,	p,	p
$^{**}4p+4$,	$p+2$,	$p+2$,	p
$^*4p+4$,	$p+2$,	$p+1$,	$p+1$
$^*4p+3$,	$p+2$,	$p+2$,	$p+1$
$^{**}4p+2$,	$p+2$,	$p+2$,	$p+2$
$3p+6$,	$2p+2$,	p,	p
$^*3p+6$,	$2p+1$,	$p+1$,	p
$^\dagger 3p+6$,	$2p$,	$p+2$,	p
$3p+6$,	$2p$,	$p+1$,	$p+1$
$^*3p+3$,	$2p+4$,	$p+1$,	p
$^*3p+3$,	$2p+2$,	$p+2$,	$p+1$
$3p+3$,	$2p+1$,	$p+2$,	$p+2$
$^\dagger 3p+2$,	$2p+4$,	$p+2$,	p

$3p+2$,	$2p+4$,	$p+1$,	$p+1$		
**$3p+2$,	$2p+2$,	$p+2$,	$p+2$		
*$3p+1$,	$2p+4$,	$p+2$,	$p+1$		
$3p$,	$2p+4$,	$p+2$,	$p+2$		
$2p+4$,	$2p+4$,	$2p$,	p		
$2p+4$,	$2p+2$,	$2p+2$,	p		
*$2p+4$,	$2p+2$,	$2p+1$,	$p+1$		
†$2p+4$,	$2p+2$,	$2p$,	$p+2$		
*$2p+2$,	$2p+2$,	$2p+2$,	$p+2$		
†$3p+6$,	$p+2$,	p,	p,	p	
$3p+6$,	$p+1$,	$p+1$,	p,	p	
*$3p+3$,	$p+2$,	$p+2$,	$p+1$,	p	
‡$3p+3$,	$p+2$,	$p+1$,	$p+1$,	$p+1$	
**$3p+2$,	$p+2$,	$p+2$,	$p+2$,	p	
**$3p+2$,	$p+2$,	$p+2$,	$p+1$,	$p+1$	
*$3p+1$,	$p+2$,	$p+2$,	$p+2$,	$p+1$	
**$3p$,	$p+2$,	$p+2$,	$p+2$,	$p+2$	
$2p+4$,	$2p+4$,	p,	p,	p	
†$2p+4$,	$2p+2$,	$p+2$,	p,	p	
$2p+4$,	$2p+2$,	$p+1$,	$p+1$,	p	
*$2p+4$,	$2p+1$,	$p+2$,	$p+1$,	p	
$2p+4$,	$2p+1$,	$p+1$,	$p+1$,	$p+1$	
‡$2p+4$,	$2p$,	$p+2$,	$p+2$,	p	
†$2p+4$,	$2p$,	$p+2$,	$p+1$,	$p+1$	
**$2p+2$,	$2p+2$,	$p+2$,	$p+2$,	p	
*$2p+2$,	$2p+2$,	$p+2$,	$p+1$,	$p+1$	
*$2p+2$,	$2p+1$,	$p+2$,	$p+2$,	$p+1$	
**$2p+2$,	$2p$,	$p+2$,	$p+2$,	$p+2$	
**$2p+1$,	$2p+1$,	$p+2$,	$p+2$,	$p+2$	
‡$2p+4$,	$p+2$,	$p+2$,	p,	p,	p
†$2p+4$,	$p+2$,	$p+1$,	$p+1$,	p,	p
$2p+4$,	$p+1$,	$p+1$,	$p+1$,	$p+1$,	p
**$2p+2$,	$p+2$,	$p+2$,	$p+2$,	p,	p
**$2p+2$,	$p+2$,	$p+2$,	$p+1$,	$p+1$,	p
**$2p+2$,	$p+2$,	$p+1$,	$p+1$,	$p+1$,	$p+1$
*$2p+1$,	$p+2$,	$p+2$,	$p+2$,	$p+1$,	p
**$2p+1$,	$p+2$,	$p+2$,	$p+1$,	$p+1$,	$p+1$
**$2p$,	$p+2$,	$p+2$,	$p+2$,	$p+2$,	p
**$2p$,	$p+2$,	$p+2$,	$p+2$,	$p+1$,	$p+1$.

First strike out all the partitions of the degree of F which are impossible

by Theorem 1. A partition containing a constituent of degree $5p+6$ is included in this category, for such a constituent is doubly transitive (II, p. 147). Likewise eliminate all those partitions which contradict Theorems 4 and 5 and those which cause a constituent group of F to contain a negative substitution. Then of the original 75 partitions of the degree of F, only 25 remain to be considered.

Now consider the possible J groups. If $d=3$, $k=3$, and J_1' can be of order 2 only, but none of the partitions that remain allow J_1' to be of order 2. Then $d=1$, $k=9$, and $9k'=18, 36, 72, 144, 360, 720, 2520, 5040$. The orders 720, 2520, and 5040 are impossible, for there are no groups of orders 80, 280, or 560 on <8 letters. Odd values of k' have not been considered, for the partitions of the degree of F show that J_1' is of even order. Moreover, no partition allows J_1' to be of order 2 or 4.

If $9k'=72$, there is only one group of order 72 on <8 letters which contains no invariant subgroup in its subgroup of order 8. The group $\{ab, ac, de, df, ad\cdot be\cdot cf\}$ with respect to one of its octic subgroups gives the J_1'

$$\begin{array}{c} 1 \\ 2437 \cdot 5698 \\ 2734 \cdot 5896 \\ 23 \cdot 47 \cdot 59 \cdot 68 \\ 47 \cdot 58 \cdot 69 \\ 27 \cdot 34 \cdot 59 \\ 24 \cdot 37 \cdot 68 \\ 23 \cdot 56 \cdot 89. \end{array}$$

The possible partitions of the degree of F for such a J_1' are the following: $5p+4, 2p+4$; $4p+4, 2p+4, p$; $3p, 2p+4, 2p+4$; $3p, 2p+4, p+2, p+2$; $2p+4, 2p+4, 2p, p$; $2p+4, 2p+2, 2p+2, p$; $2p+4, 2p+4, p, p, p$. The J groups of the partitions $3p, 2p+4, p+2, p+2$, and $2p+4, 2p+2, 2p+2, p$, are in multiple isomorphism while the constituents of J_1' are in simple isomorphism.

Now consider the partition $5p+4, 2p+4$. The subgroup L of H_{r+1} has the same transitive constituents. We shall now apply Theorem 6 to the constituent of degree $2p+4$. We know (IV, §3) that the subgroup that fixes one letter of the constituent of degree $2p+4$ has a transitive constituent of degree $2p+2$. Thus $k_1=1$, $k_2=2p+2$, and $r=5p+4$. Thus $s=(2p+4)/(5p+4)$ or $(2p+4)(2p+2)/(5p+4)$. A moment's calculation shows that these are impossible equations, for s is an integer.

In an imprimitive constituent of degree $2p+4$, generated by substitutions of order p and of degree $2p$, the invariant substitution in its J group fixes the $2p$ letters of A_1 (IV, §3). Consequently the substitution from the I

group of another constituent, associated with it, cannot be a substitution from the metacyclic group, for the substitutions from the metacyclic group have cycles on letters of each cycle of A_1. Now the I group of a constituent of degree p in F is the metacyclic group or one of its subgroups. Then in the partition $2p+4$, $2p+4$, p, p, p, F has a substitution of order 2 and degree 8. There may be two kinds of substitutions in the I group of a constituent of degree $2p$ or $3p$ in F: substitutions from the metacyclic group which do not permute cycles of A_1 and substitutions that permute cycles of A_1. The latter may again be of two kinds: substitutions which are commutative with each substitution in A_1 and substitutions which are the product of these and substitutions from the metacyclic group. The latter substitutions have cycles on letters of each cycle of A_1. Thus in the partition $2p+4$, $2p+4$, $2p$, p, the invariant substitution of order 2 and degree 8 in J_1' either fixes the $2p$ letters of the constituent of degree $2p$ or is associated with a substitution of order 2 and degree $2p$ from it. The latter is a negative substitution, while the constituent groups of F are positive groups. In the partition $2p+4$, $2p+4$, $3p$, there is likewise a substitution of order 2 and degree 8 or $8+2p$, for the invariant substitution of degree 8 may fix the $3p$ letters of A_1 in the constituent of degree $3p$, be associated with a substitution of order 2 and degree $2p$, or be associated with a substitution of order 3 and of degree $3p$ from it. Thus this partition, $2p+4$, $2p+4$, $3p$, is also impossible.

The systems of imprimitivity of a constituent of degree $2p+4$ can be chosen in only one way and the choice is determined by the transpositions in its octic J group. Then the non-invariant substitutions of order 2 and of degree 4 in the octic group permute systems of imprimitivity. Such substitutions cannot fix the $2p$ letters of A_1, for then the primitive group according to which the constituent of degree $2p+4$ permutes its systems contains a transposition and is consequently symmetric, but since the constituent groups of F are positive groups, the group of the systems is also positive. Again, since the constituent groups of F are positive groups, a positive permutation must be associated with the non-invariant substitutions of the axial subgroup of the J group. The only positive substitutions in the I group of a constituent of degree $2p+4$, on the letters of A_1 only, are substitutions from the metacyclic group. Then there must be substitutions from the metacyclic group in the I group of a constituent of degree $2p+4$. Now consider the partition $4p+4$, $2p+4$, p. The group L has transitive constituents of the same degrees. In it the constituent of degree p is simply transitive, for it cannot be doubly transitive by Theorem 1. Consequently it is a subgroup of the metacyclic group.* Since the metacyclic group has only one

* W. Burnside, Proceedings of the London Mathematical Society, vol. 33, pp. 162–185

subgroup of order p, the constituent of degree p in F is cyclic. Therefore there are no substitutions from the metacyclic group in F, for as we have seen, a permutation from the metacyclic group in I has cycles on letters of each cycle of A_1.

Let $9k' = 144$. There is only one group of order 144 on <8 letters, the group $\{abc, ad, ef, eg\}$. However, it contains no subgroup of order 16 which does not contain the axial group, an invariant subgroup of the group of order 144.

If $9k' = 360$, there is no representation, for the alternating group of degree 6 contains no subgroup of order 40.

14. Let F be of degree $7p+7$. The partitions of the degree of F are the following:

*$6p+6$,	$p+1$		
*$6p+5$,	$p+2$		
*$5p+6$,	$2p+1$		
$5p+5$,	$2p+2$		
††$5p+3$,	$2p+4$		
$4p+4$,	$3p+3$		
††$4p+1$,	$3p+6$		
*$5p+6$,	$p+1$,	p	
†$5p+5$,	$p+2$,	p	
$5p+5$,	$p+1$,	$p+1$	
*$5p+4$,	$p+2$,	$p+1$	
**$5p+3$,	$p+2$,	$p+2$	
*$4p+4$,	$2p+2$,	$p+1$	
*$4p+4$,	$2p+1$,	$p+2$	
††$4p+3$,	$2p+4$,	p	
*$4p+3$,	$2p+2$,	$p+2$	
*$4p+2$,	$2p+4$,	$p+1$	
†$4p+1$,	$2p+4$,	$p+2$	
††$3p+6$,	$3p+1$,	p	
*$3p+3$,	$3p+3$,	$p+1$	
††$3p+3$,	$3p+2$,	$p+2$	
††$3p+6$,	$2p+1$,	$2p$	
††$3p+3$,	$2p+4$,	$2p$	
††$3p+3$,	$2p+2$,	$2p+2$	
$3p+2$,	$2p+4$,	$2p+1$	
$3p+1$,	$2p+4$,	$2p+2$	
*$4p+4$,	$p+2$,	$p+1$,	p
$4p+4$,	$p+1$,	$p+1$,	$p+1$

*$4p+3$,	$p+2$,	$p+1$,	$p+1$	
*$4p+2$,	$p+2$,	$p+2$,	$p+1$	
**$4p+1$,	$p+2$,	$p+2$,	$p+2$	
††$3p+6$,	$2p+1$,	p,	p	
††$3p+6$,	$2p$,	$p+1$,	p	
††$3p+3$,	$2p+4$,	p,	p	
††$3p+3$,	$2p+2$,	$p+2$,	p	
††$3p+3$,	$2p+2$,	$p+1$,	$p+1$	
*$3p+3$,	$2p+1$,	$p+2$,	$p+1$	
††$3p+3$,	$2p$,	$p+2$,	$p+2$	
*$3p+2$,	$2p+4$,	$p+1$,	p	
*$3p+2$,	$2p+2$,	$p+2$,	$p+1$	
**$3p+2$,	$2p+1$,	$p+2$,	$p+2$	
†$3p+1$,	$2p+4$,	$p+2$,	p	
$3p+1$,	$2p+4$,	$p+1$,	$p+1$	
**$3p+1$,	$2p+2$,	$p+2$,	$p+2$	
†$3p$,	$2p+4$,	$p+2$,	$p+1$	
$2p+4$,	$2p+2$,	$2p+1$,	p	
$2p+4$,	$2p+2$,	$2p$,	$p+1$	
*$2p+4$,	$2p+1$,	$2p+1$,	$p+1$	
†$2p+4$,	$2p+1$,	$2p$,	$p+2$	
*$2p+2$,	$2p+2$,	$2p+2$,	$p+1$	
*$2p+2$,	$2p+2$,	$2p+1$,	$p+2$	
††$3p+6$,	$p+1$,	p,	p,	p
††$3p+3$,	$p+2$,	$p+2$,	p,	p
††$3p+3$,	$p+2$,	$p+1$,	$p+1$,	p
††$3p+3$,	$p+1$,	$p+1$,	$p+1$,	$p+1$
*$3p+2$,	$p+2$,	$p+2$,	$p+1$,	p
**$3p+2$,	$p+2$,	$p+1$,	$p+1$,	$p+1$
**$3p+1$,	$p+2$,	$p+2$,	$p+2$,	p
**$3p+1$,	$p+2$,	$p+2$,	$p+1$,	$p+1$
**$3p$,	$p+2$,	$p+2$,	$p+2$,	$p+1$
$2p+4$,	$2p+2$,	$p+1$,	p,	p
†$2p+4$,	$2p+1$,	$p+2$,	p,	p
$2p+4$,	$2p+1$,	$p+1$,	$p+1$,	p
†$2p+4$,	$2p$,	$p+2$,	$p+1$,	p
$2p+4$,	$2p$,	$p+1$,	$p+1$,	$p+1$
*$2p+2$,	$2p+2$,	$p+2$,	$p+1$,	p
$2p+2$,	$2p+2$,	$p+1$,	$p+1$,	$p+1$
**$2p+2$,	$2p+1$,	$p+2$,	$p+2$,	p

*$2p+2$,	$2p+1$,	$p+2$,	$p+1$,	$p+1$	
**$2p+2$,	$2p$,	$p+2$,	$p+2$,	$p+1$	
*$2p+1$,	$2p+1$,	$p+2$,	$p+2$,	$p+1$	
**$2p+1$,	$2p$,	$p+2$,	$p+2$,	$p+2$	
†$2p+4$,	$p+2$,	$p+1$,	p,	p,	p
$2p+4$,	$p+1$,	$p+1$,	$p+1$,	p,	p
**$2p+2$,	$p+2$,	$p+2$,	$p+1$,	p,	p
**$2p+2$,	$p+2$,	$p+1$,	$p+1$,	$p+1$,	p
$2p+2$,	$p+1$,	$p+1$,	$p+1$,	$p+1$,	$p+1$
**$2p+1$,	$p+2$,	$p+2$,	$p+2$,	p,	p
**$2p+1$,	$p+2$,	$p+2$,	$p+1$,	$p+1$,	p
**$2p+1$,	$p+2$,	$p+1$,	$p+1$,	$p+1$,	$p+1$
**$2p$,	$p+2$,	$p+2$,	$p+2$,	$p+1$,	p
**$2p$,	$p+2$,	$p+2$,	$p+1$,	$p+1$,	$p+1$.

In addition to striking out all those partitions of the degree of F which are impossible by Theorems 1, 4, and 5, we shall also exclude all those which bring a circular substitution of degree 3 into J_1'. For, if J_1 contains such a substitution, it is imprimitive of order 288, 576, or 1152. All groups of these orders occur for the first time on 8 letters. These partitions will be preceded by the two daggers ††. The partitions $5p+5$, $2p+2$, and $5p+5$, $p+1$, $p+1$ are also impossible, for if J_1 contains a substitution of degree and order 5 it is alternating, but J_1 is not alternating when its degree exceeds q.

The 13 partitions of the degree of F that need still to be considered show that the order of J_1' is even. Then if $d=1$, $k=8$, and $8k'=16$, 48, 144, 240, 720, 5040. The only possible orders are 16, 48, and 144, for there are no groups of orders 30, 90, and 630 on <8 letters. Moreover, the order 16 is also impossible, for all of the partitions which allow J_1' to be of order 2 have a multiply transitive constituent in L.

There are three groups of order 48 on <8 letters, namely, $\{ad, ab\cdot de, ac\cdot df\}$, $\{abc, ad, ef\}$, and $\{ab, ac\cdot bd, ef, eg\}$. The last group contains no non-invariant subgroup of order 6, while the first two give a J_1' with two transitive constituents of degree 3 each. Such constituents are incompatible with any of the partitions that remain to be considered.

The only group of order 144 on <8 letters, $\{abc, ad, ef, eg\}$, contains no subgroup of order 18 which includes no non-invariant subgroup.

If $d=2$, $k=4$, and J_1' can be of order 2 only. If $d=4$, $k=2$, and $k'=1$, but the partitions of the degree of F show that $k'\neq 1$.

15. If H_{r+1} is primitive of degree $7p+7$, it may lead to a doubly transitive group of degree $7p+8$. Then J_1 is transitive of degree 7 and J_2 is multiply transitive of degree 8. As we have seen (§3), J_2 must be the simple group of

order 168. Then if J_2 exists, J_1' is intransitive with two cyclic constituents of degree 3 each. Now F is of degree $7p+6$. The only partitions of its degree compatible with J_1' are the following: $4p+3$, $3p+3$; $3p+3$, $3p+3$, p; $p+1$, $p+1$, $p+1$, $p+1$, $p+1$, $p+1$, p. Any partition containing a constituent of degree $2p+3$ or $p+3$ is impossible, for the J groups of such constituents are of order 6.

Now consider the partition $4p+3$, $3p+3$. The constituent of degree $4p+3$ is a simply transitive primitive group (Theorem 1). Its subgroup that fixes one letter has the following partitions of its degree: $3p+2$, p; $3p+1$, $p+1$; $3p$, $p+2$; $2p+2$, $2p$; $2p+1$, $2p+1$; $2p+2$, p, p; $2p+1$, $p+1$, p; $2p$, $p+2$, p; $2p$, $p+1$, $p+1$; $p+2$, p, p, p; $p+1$, $p+1$, p, p. Since the J group of the constituent of degree $4p+3$ must be cyclic, all the partitions which bring a transposition into it are impossible. Theorem 1 excludes the following partitions: $3p+1$, $p+1$; $2p+1$, $p+1$, p; $p+1$, $p+1$, p, p. This leaves the two partitions $2p+1$, $2p+1$, and $2p$, $p+1$, $p+1$. Now apply Theorem 6 to the partition $4p+3$, $3p+3$. Let the constituent of degree $4p+3$ be the constituent of degree m. Then $k_i=2p+1$, $2p$, or $p+1$, and $r=3p+3$. Thus $s=(4p+3)(2p+1)/(3p+3)$, $(4p+3)(2p)/(3p+3)$, or $(4p+3)(p+1)/(3p+3)$. Since s is an integer all of these equations are impossible.

Theorem 6 will also be used to eliminate the partitions $3p+3$, $3p+3$, p, and $p+1$, $p+1$, $p+1$, $p+1$, $p+1$, $p+1$, p. The group J_1' demands that for these partitions L have constituents of degrees $3p+3$, $3p+3$, p. Now we know that a single constituent of degree p in L is a subgroup of the metacyclic group (§13, paragraph 7). Let the constituent of degree p be the constituent of degree m of Theorem 6. Then $k_i=b$, say, is a divisor of $p-1$, and $r=3p+3$. Thus $s=pb/(3p+3)$. Again since s is an integer this is an impossible equation.

Thus a primitive H_{r+1} of degree $7p+7$ cannot lead to a doubly transitive group of degree $7p+8$.

16. We shall now take up the case $p=11$. Suppose that F has an alternating constituent. Then the subgroup E_1 (III, §30) exists. Now H_{r+1} is of degree not greater than $7p+7$ except when E_1 has a transitive constituent simply isomorphic to its alternating constituent of degree p (compare footnote of §4). In this case E_1 has exactly three constituents of degrees p, p, $p(p-1)/2$. Then H_{r+1} is of degree not greater than $7p+14$, and F has two transitive constituents. An alternating constituent of F cannot involve letters of more than one cycle of A_1(III, §28). Then F has a transitive constituent of degree $6p+k$, $k=12$ or $\leqq 6$, and one of degree p, $p+1$, or $p+2$. This consideration eliminates F of degree $7p+13$, $7p+12$, $7p+11$, $7p+10$, and $7p+9$. Then if F is of degree $7p+8$, the only possible partition of its

degree is $6p+6$, $p+2$. If F is of degree $7p+7$, the only possible partitions of its degree are $6p+6$, $p+1$, and $6p+5$, $p+2$. These three partitions are immediately impossible by Theorem 1. Thus if F has an alternating constituent, the degree of H_{r+1} does not exceed $7p+7$. Moreover, H_{r+1} of degree $7p+7$ cannot lead to a doubly transitive group of degree $7p+8$, for the reasoning used in §15 depended in no way upon the value of p.

The partitions of the degree of F are then the same as when p was assumed >11. Now Theorem 5 was the only theorem used in eliminating partitions which depended upon the value of p. However, all the partitions eliminated by this theorem contain a constituent of degree $p+2$ and a constituent of degree $mp+n(2\leqq m\leqq 5,\ 0\leqq n<13)$. Since $p+2$ is the prime number 13, the partitions of degree $mp+n$ contain a substitution of order 13 and of degree $13m-13$ at most. Such a substitution does not respect systems of imprimitivity if the constituent is imprimitive. Then the constituents of degree $mp+n$ are primitive. Moreover, they are alternating (see theorems quoted in §1). However, if F has an alternating constituent which involves more than one cycle of A_1, G contains a substitution of order 11 and of degree <77 (III, §28). Hence these partitions are also impossible when $p=11$.

Now H_{r+1} of degree $7p+11$ was shown to be impossible (§10) except when $p=11$. We shall now consider this case. The partitions of the degree of F are the following:

$5p+10$,	$2p$		
$5p+6$,	$2p+4$		
$4p+8$,	$3p+2$		
$4p+4$,	$3p+6$		
$5p+10$,	p,	p	
$5p+6$,	$p+2$,	$p+2$	
$4p+8$,	$2p+2$,	p	
$4p+8$,	$2p+1$,	$p+1$	
$4p+8$,	$2p$,	$p+2$	
$4p+4$,	$2p+4$,	$p+2$	
$3p+6$,	$3p+3$,	$p+1$	
$3p+6$,	$3p+2$,	$p+2$	
$3p+6$,	$2p+4$,	$2p$	
$3p+6$,	$2p+2$,	$2p+2$	
$3p+2$,	$2p+4$,	$2p+4$	
$4p+8$,	$p+2$,	p,	p
$4p+8$,	$p+1$,	$p+1$,	p
$4p+4$,	$p+2$,	$p+2$,	$p+2$
$3p+6$,	$2p+4$,	p,	p

$3p+6$,	$2p+2$,	$p+2$,	p		
$3p+6$,	$2p+2$,	$p+1$,	$p+1$		
$3p+6$,	$2p+1$,	$p+2$,	$p+1$		
$3p+6$,	$2p$,	$p+2$,	$p+2$		
$3p+3$,	$2p+4$,	$p+2$,	$p+1$		
$3p+2$,	$2p+4$,	$p+2$,	$p+2$		
$2p+4$,	$2p+4$,	$2p+2$,	p		
$2p+4$,	$2p+4$,	$2p+1$,	$p+1$		
$2p+4$,	$2p+4$,	$2p$,	$p+2$		
$2p+4$,	$2p+2$,	$2p+2$,	$p+2$		
$3p+6$,	$p+2$,	$p+2$,	p,	p	
$3p+6$,	$p+2$,	$p+1$,	$p+1$,	p	
$3p+6$,	$p+1$,	$p+1$,	$p+1$,	$p+1$	
$3p+3$,	$p+2$,	$p+2$,	$p+2$,	$p+1$	
$3p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+2$	
$2p+4$,	$2p+4$,	$p+2$,	p,	p	
$2p+4$,	$2p+4$,	$p+1$,	$p+1$,	p	
$2p+4$,	$2p+2$,	$p+2$,	$p+2$,	p	
$2p+4$,	$2p+2$,	$p+2$,	$p+1$,	$p+1$	
$2p+4$,	$2p+1$,	$p+2$,	$p+2$,	$p+1$	
$2p+4$,	$2p$,	$p+2$,	$p+2$,	$p+2$	
$2p+2$,	$2p+2$,	$p+2$,	$p+2$,	$p+2$	
$2p+4$,	$p+2$,	$p+2$,	$p+2$,	p,	p
$2p+4$,	$p+2$,	$p+2$,	$p+1$,	$p+1$,	p
$2p+4$,	$p+2$,	$p+1$,	$p+1$,	$p+1$,	$p+1$
$2p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+2$,	p
$2p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+1$,	$p+1$
$2p+1$,	$p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+1$
$2p$,	$p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+2$.

Since p^2 does not divide the order of F, J_1 is transitive of degree 11. Now the largest group of order $p^q(p-1)(q!)$ on the same letters in which $\{A_1\}$ is invariant has just one subgroup of order p^q (III, §22). Then J_1 has an invariant subgroup of degree and order p and consequently is of class $p-1=10$. Consider the partitions of the degree of F. The J group of a constituent of degree $5p+10$ is of class 5 at most (IV, §3). Thus the only possible partitions are the following: $3p+2$, $p+2$, $p+2$, $p+2$, $p+2$; $2p+2$, $2p+2$, $p+2$, $p+2$, $p+2$; $2p+2$, $p+2$, $p+2$, $p+2$, $p+2$, p; $2p$, $p+2$, $p+2$, $p+2$, $p+2$, $p+2$. However, since $p+2$ is the prime number 13, all of these partitions bring a substitution of order 11 and of degree < 77 into G.

This completes the proof of the case $q=7$. It has been shown that H_{r+1}

of degree $>7p+7$ $(p>7)$, does not exist. Moreover, H_{r+1} of degree $7p+7$ can lead to a doubly transitive group of degree $7p+8$ only if it is imprimitive. Thus the degree of G cannot exceed $7p+8$.

17. It will now be shown that the degree of a primitive group of class >3, which contains a substitution of prime order $p(p>7)$ and of degree $6p$ cannot exceed $6p+6$. The present limit of $6p+10$ given by Manning depends upon the possibility of the existence of a primitive H_{r+1} of degree $6p+9$ (IV, p. 73). Furthermore, H_{r+1} of this degree can exist only if the partitions of the degree F are $4p+4$, $2p+4$ or $2p+2$, $2p+2$, $p+2$, $p+2$. We find that the real difficulty lies in trying to eliminate the former partition.

Before these partitions are discussed, a correction in the list of the partitions of the degree of F should be made. The omitted partitions are the following:

F of degree $6p+11$:

$$2p+4,\quad p+2,\quad p+2,\quad p+2,\quad p+1;$$

F of degree $6p+9$:

$$\begin{array}{lllll} 2p+4, & p+2, & p+2, & p+1, & p \\ 2p+4, & p+2, & p+1, & p+1, & p+1 \\ 2p+2, & p+2, & p+2, & p+2, & p+1 \\ 2p+1, & p+2, & p+2, & p+2, & p+2; \end{array}$$

F of degree $6p+8$:

$$\begin{array}{lllll} 2p+4, & p+2, & p+2, & p, & p \\ 2p+4, & p+2, & p+1, & p+1, & p \\ 2p+4, & p+1, & p+1, & p+1, & p+1 \\ 2p+2, & p+2, & p+2, & p+2, & p \\ 2p+2, & p+2, & p+2, & p+1, & p+1 \\ 2p, & p+2, & p+2, & p+2, & p+2; \end{array}$$

F of degree $6p+7$:

$$\begin{array}{lllll} 2p+4, & p+2, & p+1, & p, & p \\ 2p+4, & p+1, & p+1, & p+1, & p \\ 2p+2, & p+2, & p+2, & p+1, & p \\ 2p+2, & p+2, & p+1, & p+1, & p+1 \\ 2p, & p+2, & p+2, & p+2, & p+1. \end{array}$$

However, all of these partitions except the two following: $2p+4$, $p+1$, $p+1$, $p+1$, p, and $2p+4$, $p+1$, $p+1$, $p+1$, $p+1$, are immediately impossible, because either F contains a negative substitution (see §7) or Theorem 5 is contradicted. The two partitions which remain are incompatible with any of the J groups for H_{r+1} of these degrees.

We then turn to the consideration of the partitions of the degree of F that cause difficulty when H_{r+1} is of degree $6p+9$. There is an incorrect

statement (IV, p. 72) regarding the partition $3p+2$, $p+2$, $p+2$, $p+2$. However, it may immediately be dismissed from the discussion for it contradicts Theorem 5. Similarly, the partition $2p+2$, $2p+2$, $p+2$, $p+2$ is incompatible with Theorem 5. Then we need to consider only the partition $4p+4$, $2p+4$.

We recall that the only J_1' compatible with this partition is the octic group written out in §13 of this paper. Now apply Theorem 6 to this partition. Let $L(x)$ be the subgroup that fixes the letter x of H_{r+1}. Choose the constituent of degree $2p+4$ as the constituent of degree m on the letters $a_1, a_2, \cdots, a_{2p+4}$. We know (IV, §3) that $k_1=1$, and $k_2=2p+2$. Now $r=4p+4$. Then $s=(2p+4)/(4p+4)$ or $(2p+4)(2p+2)/(4p+4)$. Since s is an integer, the former of these two equations is impossible, and from the latter $s=p+2$. Thus x belongs to a transitive constituent of degree $p+2$ in $L(a_1)(a_2)$. Now $L(a_1)(a_2)$ contains substitutions of order p, for it is the subgroup that fixes one letter of the constituent of degree $4p+4$. Therefore the order of the constituent of degree $p+2$ is divisible by p, for if it were not, $L(a_1)(a_2)$ would contain a substitution of order p and of degree $<6p$. Thus the constituent of degree $p+2$ contributes a transposition to J_1'.

Let us see what J_1' demands of the subgroup $L(a_1)(a_2)$. First note that in this subgroup, the constituent of degree $4p+4$ in $L(a_1)$ can contribute at most a transposition to J_1'. J_1' then demands that the constituent of degree $2p+4$ contribute a substitution of order 2 and of degree 4 to it from this subgroup. Since the order of the constituent of degree $p+2$ in $L(a_1)(a_2)$ is divisible by p, the order of every transitive constituent of $L(a_1)(a_2)$ is divisible by p, for if it were not, the invariant subgroup generated by all the substitutions of order p in $L(a_1)(a_2)$ would fix the letters of the constituents whose order is not divisible by p, and this subgroup would bring a substitution of order 2 and of degree <6 into J_1'. Then the possible partitions of the degree of $L(a_1)(a_2)$ are the following: $p+2, p+2, 3p+2, p+1$; $p+2, p+2, 3p+2, p$; $p+2, p+2, 3p+1, p+2$; $p+2, p+2, 3p, p+2$; $p+2, p+2, 2p+2, 2p+1$; $p+2, p+2, 2p+2, 2p$; $p+2, p+2, 2p+2, p+1, p$; $p+2, p+2, 2p+2, p, p$; $p+2, p+2, 2p+1, p+2, p$; $p+2, p+2, 2p, p+2, p+1$; $p+2, p+2, 2p, p+2, p$; $p+2, p+2, p+2, p+1, p, p$; $p+2, p+2, p+2, p, p, p$. Now $L(a_1)(a_2)$ has an invariant subgroup of the same degree generated by all of its substitutions of order p. The transitive constituents of this invariant subgroup are positive groups. Consequently, the partitions which contain constituents of degree $p+2$ and p (or $p+1$) at the same time are impossible (see §7). Thus there are only the following partitions: $p+2, p+2, 3p+1, p+2$; $p+2, p+2, 3p, p+2$; $p+2, p+2, 2p+2, 2p+1$; $p+2, p+2, 2p+2, 2p$, to be considered.

Consider the last two partitions first. Now apply Theorem 6 to the group $L(a_1)$, and let the constituent of degree $4p+4$ be the constituent of degree m. Then $k_i=1$, $2p+2$, $2p+1$, or $2p$, and $r=2p+4$. Consequently $s=(4p+4)/(2p+4)$, $(4p+4)(2p+2)/(2p+4)$, $(4p+4)(2p+1)/(2p+4)$, or $(4p+4)\cdot(2p)/(2p+4)$. Since s is an integer all of these equations are impossible.

We also find the first two partitions to be impossible, for the subgroup that fixes one letter of the constituent of degree $4p+4$ cannot have constituents of the degrees given. We shall consider, then, a group of degree $4p+4$ whose subgroup that fixes one letter has constituents of degrees $3p+1$ and $p+2$ or of degrees $3p$ and $p+2$. Let $L(y)$ be the subgroup that fixes the letter y of the group of degree $4p+4$. Let $c_1, c_2, \cdots, c_{p+2}$ be the letters of the constituent of degree $p+2$ in $L(y)$. Then $L(y)(c_1)$ has a transitive constituent of degree $p+1$ on the letters $c_2, c_3, \cdots, c_{p+2}$. If the order of $L(y)$ is t, the order of $L(y)(c_1)$ is $t/(p+2)$ and the order of $L(y)(c_1)(c_2)$ is $t/[(p+2)(p+1)]$. In $L(c_1)$, y belongs to a transitive constituent of degree $p+2$. The $p+1$ letters $c_2, c_3, \cdots, c_{p+2}$ cannot form with y a transitive constituent of degree $p+2$ in $L(c_1)$, for, then, the group $\{L(y), L(c_1)\}$ has a transitive constituent of degree $p+3$, which brings a substitution of degree and order 3 into J_1'. Thus since the $p+1$ letters $c_2, c_3, \cdots, c_{p+2}$ cannot belong to the transitive constituent of degree $p+2$ in $L(c_1)$, they must belong to the transitive constituent of degree $3p+1$ or $3p$. Then the order of $L(c_1)(c_2)$ is $t/(3p+1)$ or $t/(3p)$ according as $L(c_1)$ has a constituent of degree $3p+1$ or $3p$. If y belongs to a transitive constituent of degree s in $L(c_1)(c_2)$, the order of $L(c_1)(c_2)(y)$ is $t/[(3p+1)(s)]$ or $t/(3ps)$. Then $s=(p+2)\cdot(p+1)/(3p+1)$ or $(p+2)(p+1)/(3p)$. However, s is an integer.

Thus it has been shown that a primitive group of class >3 which contains a substitution of prime order p $(p>7)$ and of degree $6p$ does not exist. The case $p=7$ will now be considered. Theorem 5 was the only theorem used in eliminating partitions which depended upon the value of p. The partitions thrown out by means of this theorem were $2p+1, p+2, p+2, p+2, p+2$; $3p+2, p+2, p+2, p+2$; $2p+2, 2p+2, p+2, p+2$; $2p, p+2, p+2, p+2, p+2$. Manning has already shown that the third partition is impossible when $p=7$ (IV, p. 78). In the first partition the constituent of degree $2p+1$ $(=15)$ is primitive. Moreover it is doubly transitive, for a simply transitive primitive group of degree 15 whose subgroup that fixes one letter has two transitive constituents of degree 7 does not exist.*

This partition is then impossible by Theorem 1. In the second partition the constituent of degree $3p+2=23$, a prime number. Then G has a substitu-

* G. A. Miller, Proceedings of the London Mathematical Society, vol. 28 (1897), p. 540.

tion of degree and order 23 and consequently its degree does not exceed 25. In the last partition, the constituent of degree $2p(=14)$ is imprimitive, for a simply transitive group of degree 14 does not exist,* and a multiply transitive group of degree 14 is impossible by Theorem 1. Now the constituent of degree 9 is at least triply transitive and consequently is either the Mathieu group of order 504 or the group of order 1512 and of class 6. The constituent of degree 14 has systems of imprimitivity of two letters only. Its group in the systems is a primitive group of degree 7. Then the group of degree 9 cannot be simply isomorphic to this group in the systems, for these groups of degree 9 occur for the first time on 9 letters. The Mathieu group is then impossible, for it is a simple group. The only invariant subgroup of the group of order 1512 is the Mathieu group of order 504. Then the group in the systems of the constituent of degree 14 must have a quotient group of order 3. However, since the constituent of degree 14 contains more than one subgroup of order p, its group in the systems cannot have such a quotient group.

Thus the theorem for the case $q=6$ now reads

The degree of a primitive group of class >3 which contains a substitution of prime order $p(p>7)$ and of degree $6p$ cannot exceed $6p+6$. If $p=7$, the true limit of the degree of G is $6p+7$.

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* G. A. Miller, Quarterly Journal of Mathematics, vol. 29 (1897), p. 242.

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GENERALIZED LAGRANGE PROBLEMS IN THE CALCULUS OF VARIATIONS*

BY

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I. Introduction

In the new dynamical theory of economics there arises a very general problem which can be said to be a generalization of the Lagrange problem in the calculus of variations.‡ It will not be necessary to consider the formulation of the corresponding economic theory here since I have already done this in another paper.§ It would hardly be fair, however, to introduce the reader to a rather unusual mathematical situation without giving some hint as to its origin. It seems desirable, therefore, to give first a brief economic formulation of the problem whose mathematical aspects will be discussed in this paper.

If there are two producers of an identical commodity C, manufacturing, respectively, amounts $u_1(x)$ and $u_2(x)$ of C per unit time, subject to the respective cost functions $\phi_1(u_1, u_1', u_2, u_2', u_3, u_3', x)$ and $\phi_2(u_1, u_1', u_2, u_2', u_3, u_3', x)$, where $u_3(x)$ is the selling price of C at a time x, then the respective profits obtained during an interval of time $x_0 \leqq x \leqq x_1$ are

$$I_1 = \int_{x_0}^{x_1} [u_3 u_1 - \phi_1(u_1, u_1', u_2, u_2', u_3, u_3', x)]dx,$$

$$I_2 = \int_{x_0}^{x_1} [u_3 u_2 - \phi_2(u_1, u_1', u_2, u_2', u_3, u_3', x)]dx,$$

where ϕ_1 and ϕ_2 are assumed to be continuous with their first and second derivatives with respect to all their arguments, and primes denote derivatives with respect to time x.

The rates of production $u_1(x)$ and $u_2(x)$ and the price $u_3(x)$ will satisfy an equation of demand which in the general case will be of the form

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‡ For a special example of this problem, see C. F. Roos, *A mathematical theory of competition*, American Journal of Mathematics, vol. 47 (1925), pp. 163–175. See also G. C. Evans, *The dynamics of monopoly*, American Mathematical Monthly, vol. 31 (1921).

§ C. F. Roos, *A dynamical theory of economics*, Journal of Political Economy, vol. 35 (1927). See also Roos, *Dynamical economics*, Proceedings of the National Academy of Sciences, vol. 13 (1927).

$$G(u_1, u_1', \cdots, u_3', x) = \int_{x_0}^{x} P(u_1, u_1', \cdots, u_3', x, s)ds \tag{1}$$

where G and P have continuity properties similar to those of ϕ_1 and ϕ_2.* Each manufacturer will consider his rate of production to be influenced by the rate of production of his competitor only through the equation of demand, and will desire to determine his own rate of production in such a way that he obtains a maximum profit over some interval of time, say $x_0 \leqq x \leqq x_1$.

The problem of competition for this state of affairs will then be the problem of determining a curve Γ in the space (u_1, u_2, u_3, x), satisfying a functional equation (1), such that an integral I_1, taken along Γ from x_0 to x_1, is a maximum when u_2 is momentarily held fixed, and such that a second integral I_2, also taken along Γ from x_0 to x_1, is a maximum when u_1 is momentarily held fixed. In the usual case the initial time x_0 and the corresponding initial values of the u_i, $i=1, 2, 3$, are fixed. The end time x_1 and the corresponding end values of the u_i may be regarded as fixed or not, depending upon the nature of the problem under consideration. Both cases will be considered at some length in the following paragraphs.

For the particular case $P\equiv 0$ the equation of demand becomes simply a first-order differential equation. For this case the problem of competition can be solved by the methods employed in the classical Lagrange problem in the calculus of variations.† In order to obtain a solution in the classical way we need, however, two sets of Lagrange multipliers, and this makes the problem quite difficult. In the following pages I shall give an analysis for the case in which the rates of production and price are related by a differential equation of demand $G(u_1, u_1', u_2, u_2', u_3, u_3', x)=0$ without using multipliers, and shall obtain necessary and sufficient conditions. These conditions, although functional in character, seem simpler than the corresponding conditions which would be obtained by the classical analysis.

In discussing the Lagrange problem for several differential equations $G_k(u_1, u_1', \cdots, u_n, u_n', x)=0$, $k=1, \cdots, m<n$, I introduce the theory of Volterra integral equations into my analysis to replace the classical theory by means of multipliers. This use of the theory of integral equations enables me to obtain a method for solving the more general problem for which $P_k(u_1, u_1', \cdots, u_n, u_n', x, s)\not\equiv 0$. So far as I know this use of integral equa-

* Roos, *Dynamical economics*, loc. cit.

† J. Hadamard, *Leçons sur le Calcul des Variations*, pp. 217 and sequence. See also G. A. Bliss, *The Problem of Lagrange in the Calculus of Variations*, lectures given at the University of Chicago, summer quarter 1925, mimeographed by O. E. Brown, Northwestern University, Evanston, Illinois.

tions is entirely new. As a result the following exposition, although lengthy, does not represent a complete treatment of the subject.

II. Fixed end points. Eulerian equations in functional form

1. Geometrical interpretation of the problem. In order to make our analysis easier to follow let us first examine the problem for which both end points are fixed, and for which $P(u_1, u_1', \cdots, x, s) \equiv 0$, from a geometrical view point. In the hyperspace (u_1, u_2, u_3, x) let $u_2 = u_2(x)$ be any function, continuous with its first derivative, and substitute this value of u_2 in the integrand $F_1(u_1, u_1', \cdots, u_3', x)$ of an integral I_1, corresponding to the I_1 of the introduction, and in the differential equation $G=0$. The function F_1 becomes a function $F_1(u_1, u_1', u_2(x), u_2'(x), u_3, u_3', x)$, and $G=0$ becomes a differential equation $G(u_1, u_1', u_2(x), u_2'(x), u_3, u_3', x)=0$. The problem of finding $u_1 = y_1(x)$ which maximizes I_1 is thus reduced to the problem of finding a function $y_1(x)$ which maximizes

$$I_1 = \int_{x_0}^{x_1} F_1(u_1, u_1', u_2(x), u_2'(x), u_3, u_3', x)dx,$$

and satisfies $G=0$ and given end conditions whatever they may be.

Again, if $u_1(x) = y_1(x)$ be substituted in the integrand F_2 and in $G=0$, these become, respectively, $F_1(y_1(x), y_1'(x), u_2, u_2', u_3, u_3', x)$ and $G(y_1(x), y_1'(x), u_2, u_2', u_3, u_3', x)=0$. Choosing the function $u_2(x) = y_2(x)$ so that it satisfies $G=0$ and maximizes

$$I_2 = \int_{x_0}^{x_1} F_2(y_1(x), y_1'(x), u_2, u_2', u_3, u_3', x)dx$$

completes the solution of the problem, for u_1 and u_3 have already been determined in terms of $u_2(x)$. It is important to note that we have assumed the existence of a solution without showing that one actually exists. Conditions for the existence of a solution will be discussed in Part IV of this paper.

2. Admissible arcs and variations. An arc $u_i = u_i(x)$, $i = 1, 2, 3$, which is continuous on the interval $x_0 \leqq x \leqq x_1$, and is such that the interval can be divided into a finite number of subintervals on each of which the functions $u_i(x)$ have continuous derivatives up to and including those of the second order will be called an *admissible arc*. This definition will permit a maximizing arc to have a finite number of corners. All of the elements of an admissible arc shall be required to lie in a simply connected region of a hyperspace (u_1, u_2, u_3, x), and to satisfy the differential equation $G(u_1, u_1', u_2, u_2', u_3.$

u_3', $x) = 0$, and, furthermore, to satisfy certain end conditions.* In the following paragraphs all admissible arcs will be regarded as fixed at a fixed x_0, i.e.

$$u_1(x_0) = u_{10}, \quad u_2(x_0) = u_{20}, \quad u_3(x_0) = u_{30}, \tag{2}$$

and either variable or fixed at x_1 depending upon the particular problem under consideration. The behavior of the arcs at x_1 will be pointed out as the work progresses.

If a two-parameter family of admissible arcs $u_i = u_i(x, a, b)$ containing a particular admissible arc Γ for the parametric values $a = b = 0$ be given, we shall call the functions

$$\xi_1(x) = \partial u_1(x,0,0)/\partial a, \quad \xi_2(x) = \partial u_2(x, 0, 0)/\partial b$$

partial variations of the family along Γ. Ordinarily we would require a three-parameter family to cover the space (u_1, u_2, u_3, x) completely, but the differential equation $G = 0$ and the initial condition $u_3(x_0) = u_{30}$ removes one degree of freedom.

3. The Eulerian equations in functional form. Let us write

$$\begin{aligned} u_\alpha &= y_\alpha + \psi_\alpha(x,a,b) \qquad (\alpha = 1,2), \\ u_3 &= y_3 + \theta(x,a,b), \end{aligned} \tag{3}$$

where the ψ_α are functions, continuous in x, a and b, possessing continuous derivatives of the first order with respect to x, a and b and vanishing when a and b vanish. The functions y_α and y_3 are the functions $u_i(x)$, $i = 1, 2, 3$, defining the maximizing curve Γ which we suppose for the present to exist a priori.

In our analysis we shall have to require that the functions F_1, F_2 and G possess continuous derivatives of the second order with respect to each of the arguments u_i, u_i', x, $i = 1, 2, 3$, and, furthermore, that $\partial G/\partial u_3' \neq 0$ in the interval $x_0 \leqq x \leqq x_1$. Under these hypotheses the function θ is determined by $G = 0$ and the first two equations of (3), except for an arbitrary constant, as a continuous function of x, a, b with continuous derivatives of the first order.

The derivatives $\partial\theta/\partial a$ and $\partial\theta/\partial b$ satisfy the *equations of partial variations*

$$(\partial G/\partial u_1)\partial\psi_1/\partial a + (\partial G/\partial u_1')\partial\psi_1'/\partial a + (\partial G/\partial u_3)\partial\theta/\partial a + (\partial G/\partial u_3')\partial\theta'/\partial a = 0,$$

$$(\partial G/\partial u_2)\partial\psi_2/\partial b + (\partial G/\partial u_2')\partial\psi_2'/\partial b + (\partial G/\partial u_3)\partial\theta/\partial b + (\partial G/\partial u_3')\partial\theta'/\partial b = 0,$$

and will, therefore, also be continuous and have continuous partial derivatives

* See Bliss, loc. cit., p. 3.

of the first order, on account of the continuity requirements on G. We further restrict the ψ_α by the following conditions:

$$\partial\psi_1/\partial a = \xi_1(x), \qquad \partial\psi_1/\partial b = 0,$$
$$\partial\psi_2/\partial a = 0, \qquad \partial\psi_2/\partial b = \xi_2(x),$$

when $a=b=0$. We employ the following notation: $\partial\theta/\partial a=\theta_a(x)$ and $\partial\theta/\partial b=\theta_b(x)$ when $a=b=0$.

Since we have assumed the end values of u_1 and u_2 to be fixed at x_1 as well as at x_0, we can write

$$\xi_1(x_0) = \xi_1(x_1) = \xi_2(x_0) = \xi_2(x_1) = 0.$$

For the parametric values $a=b=0$ the function $\theta(x, 0, 0)=\theta(x)$ must satisfy the differential equations of partial variations

$$(\partial G/\partial u_1)\xi_1 + (\partial G/\partial u_1')\xi_1' + (\partial G/\partial u_3)\theta_a + (\partial G/\partial u_3')\theta_a' = 0, \tag{4A}$$

$$(\partial G/\partial u_2)\xi_2 + (\partial G/\partial u_2')\xi_2' + (\partial G/\partial u_3)\theta_b + (\partial G/\partial u_3')\theta_b' = 0. \tag{4B}$$

The first of these determines θ_a in terms of ξ_1 and the partial derivatives of G with respect to u_1 and u_1', except for a constant, whereas the second determines θ_b in terms of ξ_2 and the partial derivatives of G with respect to u_2 and u_2', except for a constant. Choosing these constants so that each of the partial variations $\theta_a(x_0)$ and $\theta_b(x_0)$ vanishes implies that the total variation of the function θ be zero at x_0, i.e. $\delta\theta=\theta_a\delta a+\theta_b\delta b=0$ at $x=x_0$. Conversely, since the ψ_α are arbitrary, the vanishing of $\delta\theta$ implies the vanishing of both θ_a and θ_b. The equations (4) and the initial conditions (2), therefore, completely determine the variations of u_3. The functions u_1 and u_2 have thus been classified as independent functions in a manner similar to the way in which variables are classified in the ordinary theory of maxima and minima of functions.

If functions $u_i(x, a, b)$, defining a two-parameter family of admissible arcs containing Γ for the parametric values $a=b=0$, are substituted in I_1. this integral becomes a function of a and b defined by

$$I_1(a,b) = \int_{x_0}^{x_1} F_1(u_1(x,a,b), u_1'(x,a,b), \cdots, u_3'(x,a,b), x)dx.$$

The partial variation of this integral with respect to a reduces to

$$(\partial I_1/\partial a)\delta a = \int_{x_0}^{x_1}[(\partial F_1/\partial y_1)\xi_1 + (\partial F_1/\partial y_1')\xi_1' + (\partial F_1/\partial y_3)\theta_a + (\partial F_1/\partial y_3')\theta_a']dx\delta a$$

for $a=b=0$.

Instead of proceeding in the classical way we shall solve the differential equation (4) for θ_a and develop a theory without the use of Lagrange multipliers.* This procedure seems to be more directly an extension of the ordinary theory of maxima and minima; it allows us to obtain the Weierstrass, Legendre and Jacobi conditions by an analysis which is simpler than that used in the classical theory, and, furthermore, it leads to a method for solving the Lagrange problem when the differential equations are replaced by functional equations of the type (1). We proceed as follows:

Since by hypothesis $\partial G/\partial y_3'$ is not zero in the interval $x_0 \leqq x \leqq x_1$, the solution of (4) for θ_a is

$$\theta_a = \int_{x_0}^{x} e^{V_3}[(\partial G_3'/\partial y_1)\xi_1 + (\partial G_3'/\partial y_1')\xi_1']dt, \tag{5}$$

where the following notation has been introduced: $(\partial G/\partial y_3)/(\partial G/\partial y_3') = -\partial G_3'/\partial y_3$; $(\partial G/\partial y_1)/(\partial G/\partial y_3') = -\partial G_3'/\partial y_1$; $(\partial G/\partial y_1')/(\partial G/\partial y_3') = -\partial G_3'/\partial y_1'$; $V_3 = \int_t^x (\partial G_3'/\partial y_3)ds$. The assumption $\theta_a(x_0) = 0$, made above, does not necessarily impose a limitation on this method, for, if u_3 were variable at x_0, the solution for θ_a would be the solution above plus the variation of u_3 at x_0.

Differentiation of (5) with respect to x determines θ_a' by the formula

$$\begin{aligned} \theta_a' = {} & (\partial G_3'/\partial y_1)\xi_1 + (\partial G_3'/\partial y_1')\xi_1' \\ & + (\partial G_3'/\partial y_3)\int_{x_0}^{x} e^{V_3}[(\partial G_3'/\partial y_1)\xi_1 + (\partial G_3'/\partial y_1')\xi_1']dt. \end{aligned} \tag{6}$$

When the values of θ_a and θ_a' as given by (5) and (6) are substituted in the expression defining the partial variation of I_1 with respect to a, it becomes for $a=b=0$

$$\begin{aligned} (\partial I_1/\partial a)\delta a = \int_{x_0}^{x_1}\Big[& [\partial F_1/\partial y_1 + (\partial F_1/\partial y_3')\partial G_3'/\partial y_1]\xi_1 + [(\partial F_1/\partial y_3')\partial G_3'/\partial y_1' \\ & + \partial F_1/\partial y_1']\xi_1' + [\partial F_1/\partial y_3 + (\partial F_1/\partial y_3')\partial G_3'/\partial y_3]\int_{x_0}^{x} e^{V_3}[(\partial G_3'/\partial y_1)\xi_1 \\ & + (\partial G_3'/\partial y_1')\xi_1']dt\Big]dx. \end{aligned}$$

An application of Dirichlet's formula for changing the order of integration of an iterated integral, followed by an interchange of t and x, the parameters of integration, yields the equation

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* Hadamard, loc. cit., Chapter VI, gives the classical theory.

$$(\partial I_1/\partial a)\delta a = \int_{x_0}^{x_1}\Big[[\partial F_1/\partial y_1 + (\partial F_1/\partial y_3')\partial G_3'/\partial y_1 + (\partial G_3'/\partial y_1)W_1]\xi_1$$
$$+ [\partial F_1/\partial y_1' + (\partial F_1/\partial y_3')\partial G_3'/\partial y_1' + (\partial G_3'/\partial y_1')W_1]\xi_1' \Big] dx,$$

where

$$W_1 = \int_x^{x_1} e^{V_3}[\partial F_1/\partial y_3 + (\partial F_1/\partial y_3')\partial G_3'/\partial y_3]dt.$$

Since $\xi_1(x)$ vanishes at x_0 and x_1 by hypothesis, an integration by parts performed on the terms involving $\xi_1(x)$ of the partial variation of I_1 with respect to a furnishes the expression

$$(\partial I_1/\partial a)\delta a = \int_{x_0}^{x_1}\Big[- \int_{x_0}^{x} [\partial F_1/\partial y_1 + (\partial F_1/\partial y_3')\partial G_3'/\partial y_1 + (\partial G_3/\partial y_1)W_1]dt$$
$$+ [\partial F_1/\partial y_1' + (\partial F_1/\partial y_3')\partial G_3'/\partial y_1' + (\partial G_3'/\partial y_1')W_1] \Big] \xi_1'(x)dx,$$

where the coefficient of $\xi_1'(x)$ is continuous because of the continuity requirements on F_1 and G.

If I_1 is to be a maximum along the curve Γ, it is necessary that $(\partial I_1/\partial a)\delta a$ be zero for all values of the functions $\xi_1(x)$. By a well known theorem of the calculus of variations it follows that the coefficient of $\xi_1'(x)$ must be a constant, that is,

$$\text{(7)}\qquad \partial F_1/\partial y_1' + (\partial F_1/\partial y_3')\partial G_3'/\partial y_1' + (\partial G_3'/\partial y_1')W_1$$
$$= \int_{x_0}^{x} [\partial F_1/\partial y_1 + (\partial F_1/\partial y_3')\partial G_3'/\partial y_1 + (\partial G_3'/\partial y_1)W_1]dt + C_1,$$

where C_1 is a constant to be determined by the initial conditions.

An entirely similar analysis applied to I_2 yields the necessary condition

$$\text{(8)}\qquad \partial F_2/\partial y_2' + (\partial F_2/\partial y_3')\partial G_3'/\partial y_2' + (\partial G_3'/\partial y_2')W_2$$
$$= \int_{x_0}^{x} [\partial F_2/\partial y_2 + (\partial F_2/\partial y_3')\partial G_3'/\partial y_2 + (\partial G_3'/\partial y_2)W_2]dt + C_2.$$

The functional-differential equations (7) and (8) are the analogues of the Euler equations in the Du Bois-Reymond form.* Wherever the maximizing curve Γ has a continuously turning tangent we can differentiate (7) and (8) with respect to x and obtain functional-differential equations which involve

* Du Bois-Reymond, Mathematische Annalen, vol. 15 (1879), p. 313.

second-order derivatives and which are the analogues of the Euler equations. We can, therefore, state the following theorem.

THEOREM 1. *In order that an admissible arc* Γ *in the space* (u_1, u_2, u_3, x), *satisfying a differential equation* $G(u_1, u_1', u_2, u_2', u_3, u_3', x)=0$ *and initial conditions* $u_i(x_0)=u_{i0}$, $u_i(x_1)=u_{i1}$, *maximize an integral* I_1 *when* u_2 *is not allowed to vary and at the same time maximize a second integral* I_2 *when* u_1 *is not allowed to vary, it is necessary that this curve satisfy the functional-differential equations* (7) *and* (8). *If the maximizing curve has a continuously turning tangent at* x, $x_0 \leqq x \leqq x_1$, *it must satisfy the equations*

$$\partial F_k/\partial y_k + (\partial F_k/\partial y_3')\partial G_3'/\partial y_k + (\partial G_3'/\partial y_k)W_k - \frac{d}{dx}[\partial F_k/\partial y_k' + (\partial F_k/\partial y_3')\partial G_3'/\partial y_k' + (\partial G_3'/\partial y_k')W_k] = 0 \quad (k = 1,2), \tag{9}$$

obtained by differentiating (7) *and* (8) *with respect to* x.

Functional-differential equations of the type (9), with our form of W_k, have not been discussed in the literature. It would be desirable to be able to say that a unique solution of these equations plus the differential equation $G=0$ exists whenever end values $u_i(x_0)=u_{i0}$ and $u_i(x_1)=u_{i1}$ are given. This problem will not be discussed in the present paper.* It may be mentioned, however, that I have already exhibited a special example for which the system (9) reduces to a system of Volterra integral equations, and have actually found the solution.† Let us examine (7) and (8) from a different point of view.

In particular if $F_1 \equiv F_2$, the problem reduces to a strict Lagrange problem. No assumption which would prevent this has been made, hence we have the following

COROLLARY. *The equations resulting from* (7) *and* (8) *by putting* $F_1 \equiv F_2$ *must be satisfied by a curve satisfying a differential equation* $G=0$ *and initial conditions* $u_i(x_0)=u_{i0}$, $u_i(x_1)=u_{i1}$ *if this curve is to maximize an integral*

$$I = \int_{x_0}^{x_1} F_1(u_1, u_1', u_2, u_2', u_3, u_3', x)dx$$

in which both u_1 *and* u_2 *vary independently.*

---

* L. M. Graves, *Implicit functions and differential equations in general analysis*, these Transactions, vol. 29, pp. 515–552, gives imbedding and existence theorems for a system which includes (9) as a special case. If y_k''' is continuous, we can reduce (9) to a differential equation of the third order by a differentiation, because of the form of W_k, and existence theorems for differential equations will apply.

† Roos, *A mathematical theory of competition*, loc. cit., p. 167.

The methods of this part can be extended without difficulty to the case for which there are n integrals

$$I_h = \int_{x_0}^{x_1} F_h(u_1, u_1', \cdots, u_n, u_n', u_{n+1}, u_{n+1}', x)dx \qquad (h = 1, 2, \cdots, n)$$

and one differential equation $G(u_1, u_1', \cdots, u_n, u_n', u_{n+1}, u_{n+1}', x) = 0$, in which case n functional equations of the type (7) result.

III. Variable end points. Analogues of Weierstrass and Legendre conditions

4. Problem with one end point variable. In the preceding paragraphs a problem in simultaneous maxima for fixed end points has been considered. The problem is even more interesting when one end parameter, say x_1, and the corresponding end values are allowed to vary.

Consider the problem of determining a curve Γ in the space (u_1, u_2, u_3, u_4, x) satisfying a differential equation

$$G(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x) = 0$$

such that an integral

$$I_1 = \int_{x_0}^{x_1} F(u_1, u_1', \cdots, u_4, u_4', x)dx$$

is a maximum when u_1 and u_2 are allowed to vary independently, but not u_3, and such that a second integral

$$I_2 = \int_{x_0}^{x_1} F_2(u_1, u_1', \cdots, u_4, u_4', x)dx$$

is a maximum when u_3 is allowed to vary independently, but not u_1 and u_2. We assume the end parameter x_0 and the end values $u_i(x_0) = u_{i0}$ to be fixed, and the end parameter x_1 and the corresponding end values of the u_i to be variable. Let us assume as we did in Part II that $\partial G/\partial u_4' \neq 0$ for the region which contains admissible arcs $u_i(x)$, $i = 1, 2, 3, 4$, and that the functions F_α, $\alpha = 1, 2$, and G are continuous in $u_1, u_2, u_3, u_4, u_1', u_2', u_3', u_4', x$ and have continuous partial derivatives of the first order with respect to these arguments.

5. Functional transversality conditions. In the functions F_α and G replace the functions $u_i(x)$, $i = 1, 2, 3, 4$, by a set $u_i = f_i(x, a_1, a_2, a_3)$, where the f_i are functions of x and parameters a_1, a_2 and a_3, continuous and admitting continuous derivatives up to the second order with respect to x and these parameters in the domain $0 \leqq a_1 \leqq h$; $0 \leqq a_2 \leqq h$; $0 \leqq a_3 \leqq h$; $x_0 \leqq x \leqq x_1$. The

functions f_1, f_2 and f_3 are otherwise arbitrary, but f_4 is determined by $G=0$ and the initial condition $u_4(x_0)=u_{40}$. Let the limit of integration x_1 be a similar function of the parameters a_σ, $\sigma=1, 2, 3$, i.e. $x_1=\phi(a_1, a_2, a_3)$.

By the ordinary rules of differentiation the differential of the integral I_1, which is also a function of the a_σ, is for u_3 ($=$constant)

$$
(10) \qquad dI_1 = [F_1(u_1,u_1', \cdots, u_4,u_4',x)\delta x]^{x_1} + \int_{x_0}^{x_1}[(\partial F_1/\partial u_i)\delta f_i+(\partial F_1/\partial u_i')\delta f_i']dx,
$$

where $\delta f_i=(\partial F_i/\partial a_1)\delta a_1+(\partial f_i/\partial a_2)\delta a_2+(\partial f_i/\partial a_3)\delta a_3$ and i is an umbral index for the values 1, 2, 4, but not for 3, according to the convention that whenever a literal suffix appears twice in a term that term is to be summed for values of the suffix.* The variation of u_3 in F_1 is by hypothesis equal to zero, hence $\delta f_3\equiv 0$.

As already stated the variations of f_1 and f_2 are to be arbitrary (except for continuity properties), but we can not take the variation of f_4 to be arbitrary, for it is determined by the differential equation of partial variation

$$
(\partial G/\partial u_4)\delta f_4 + (\partial G/\partial u_4')\delta f_4' + (\partial G/\partial u_k)\delta f_k + (\partial G/\partial u_k')\delta f_k' = 0,
$$

where k is an umbral index taking on the values 1 and 2 only. Since $\partial G/\partial u_4'$ does not vanish and is continuous in the interval by hypothesis, and, furthermore, since $\delta(df/dx)=(d/dx)\delta f$, this expression can be regarded as a first-order differential equation for the determination of δf_4 in terms of δf_1 and δf_2 and the initial value of δf_4 at $x=x_0$. Since we have supposed $\delta f_4(x_0)=0$, we may write

$$
\delta f_4 = \int_{x_0}^{x} e^{V_4}[(\partial G_4'/\partial u_k)\delta f_k + (\partial G_4'/\partial u_k')\delta f_k']dt,
$$

where the expressions of the form $\partial G_4'/\partial u_k$, etc. have meanings similar to the corresponding ratios defined in (5). As in (5) we determine the value of $\delta f_4'$ by differentiation of the above expression. If the values of δf_4 and $\delta f_4'$ so found be substituted in (10), it becomes

$$
\begin{aligned}
F_1\delta x]^{x_1} + \int_{x_0}^{x_1}\Big[& [\partial F_1/\partial u_k + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k)]\delta f_k \\
& + [\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k')]\delta f_k' \\
+ [\partial F_1/\partial u_4+(\partial F_1/\partial u_4')(\partial G_4'/\partial u_4)] & \int_{x_0}^{x} e^{V_4}[(\partial G_4'/\partial u_k)\delta f_k+(\partial G_4'/\partial u_k')\delta f_k']dt\Big]dx.
\end{aligned}
$$

---

* See A. S. Eddington, *The Mathematical Theory of Relativity*, p. 50.

An application of Dirichlet's formula to the iterated integral followed by an interchange of the parameters x and t as before reduces the above formula for δI_1 to

$$\text{(10B)}\quad \delta I_1 = F_1\delta x]^{x_1} + \int_{x_0}^{x_1}\Big[[\partial F_1/\partial u_k + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k) + (\partial G_4'/\partial u_k)W_1]\delta f_k + [\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k') + (\partial G_4'/\partial u_k')W_1]\delta f_k'\Big]dx,$$

where

$$W_1 = \int_{x_1}^{x} e^{V_4}[(\partial F_1/\partial u_4')(\partial G_4'/\partial u_4) + \partial F_1/\partial u_4]dt.$$

Since the $f_k(x, a_1, a_2, a_3)$, $k=1, 2$, have by hypothesis continuous second derivatives with respect to x, the formula for integration by parts can be applied to the second member of (10B), so that*

$$\delta I_1 = \Big[F_1\delta x + [\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k') + (\partial G_4'/\partial u_k')W_1]\delta f_k\Big]_{x_0}^{x_1} + \int_{x_0}^{x_1}\Big[\partial F_1/\partial u_k + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k) + (\partial G_4'/\partial u_k)W_1 - \frac{d}{dx}[\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k') + (\partial G_4'/\partial u_k')W_1]\Big]\delta f_k dx,$$

where k is umbral as before.

By definition $\delta f_k = (\partial F_k/\partial a_i)\delta a_i$, where i is umbral with range 1, 2, 3, hence the variation of u_k is given by $\delta u_k = u_k'\delta x + \delta f_k$. If the value of δf_k defined by this equation be substituted in δI_1, the following formula results:

$$\delta I_1 = \Big[\partial F_1/\partial u_k' + [(\partial F_1/\partial u_4')(\partial G_4'/\partial u_k') + (\partial G_4'/\partial u_k')W_1][\delta u_k - u_k'\delta x]\Big]_{x_0}^{x_1} + \int_{x_0}^{x_1}\Big[\partial F_1/\partial u_k + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k) + (\partial G_4'/\partial u_k)W_1 - \frac{d}{dx}[\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k') + (\partial G_4'/\partial u_k')W_1]\Big]\delta f_k dx.$$

We define an arc $u_i = u_i(x)$, $i=1, 2, 3, 4$, as an *extremal arc* if it has continuous derivatives du_i/dx and d^2u_i/dx^2 in the interval $x_0 \leqq x \leqq x_1$, and if, furthermore, it satisfies the differential equation $G(u_1, u_1', \cdots, u_4, u_4', x) = 0$, the set of two equations

* Hadamard, loc cit., p. 60.

$$
\begin{aligned}
(11)\qquad & \partial F_1/\partial u_k + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k) + (\partial G_4'/\partial u_k)W_1 - \frac{d}{dx}[\partial F_1/\partial u_k' \\
& + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k') + (\partial G_4'/\partial u_k')W_1] = 0 \qquad (k = 1,2),
\end{aligned}
$$

and a similar set for the integral I_2.

If an extremal Γ is to maximize I_1 for u_3 constant, it is necessary that the differential

$$\delta I_1(\Gamma) = F_1\delta x_1 + [\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')\partial G_4'/\partial u_k'][\delta u_k(x_1) - u_k'\delta x_1]$$

vanish for all possible choices of δx_1 and $\delta u_k(x_1)$, $k=1, 2$. We can thus state the transversality theorem:

THEOREM 2. *If for an admissible arc Γ, one of whose end points is fixed at x_0 while the other varies over a V_3 defined by u_3=constant and a differential equation $G=0$, the value $I_1(\Gamma)$, for u_3=constant, $G=0$, is a maximum with respect to the values of I_1 on neighboring admissible arcs, issuing from the same fixed point 0, then at the intersection point 1 of Γ with V_3, the directional coefficients of V_3 and the element $(u_1, u_1', \cdots, u_4', x)$ of Γ must satisfy the relations*

$$
(12)\qquad
\begin{aligned}
& F_1(u_1,u_1', \cdots, u_4',x) - [\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k')]u_k' = 0, \\
& \partial F_1/\partial u_i' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_i') = 0^{*} \qquad (i = 1,2).
\end{aligned}
$$

If we apply a similar analysis to the integral I_2, for u_1 and u_2 constant, we obtain a differential

$$\delta I_2(\Gamma) = F_2\delta x_1 + [\partial F_2/\partial u_3' + (\partial F_2/\partial u_3')(\partial G_4'/\partial u_3')][\delta u_3(x_1) - u_3'(x_1)\delta x_1]$$

along an extremal for the integral I_2. If, therefore, Γ is also to maximize I_2 for u_1 and u_2 constant, then at 1, the intersection of Γ with V_2, defined by u_1=constant, u_2=constant and $G=0$, it is necessary that the equations

$$
(13)\qquad
\begin{aligned}
& F_2(u_1,u_1', \cdots, u_4',x) + [\partial F_2/\partial u_3' + (\partial F_2/\partial u_4')(\partial G_4'/\partial u_3')]u_3' = 0, \\
& \partial F_2/\partial u_3' + (\partial F_2/\partial u_4')(\partial G_4'/\partial u_3') = 0
\end{aligned}
$$

---

* From a consideration of the classical theory of the Lagrange problem with second end point variable we would expect to have four transversality conditions instead of three as given by (12), but we have not used the condition that $\delta u_4(x_1)$ is arbitrary, since it is a function of arbitrary functions δf_k, and hence we lack this condition. If we perform an integration by parts on the term in $\delta f_i'$ of (10), and then substitute for δf_i as we did above, we obtain a term $(\partial F_1/\partial u_4')\delta f_4$ besides terms in δf_k. Since δf_4 is arbitrary at x_1 it follows that $\partial F_1/\partial u_4'=0$ at $x=x_1$. We may, therefore, by the help of (12) write the transversality condition as

$$
(12A)\qquad
\begin{aligned}
& F_1(u_1, u_1', \cdots, u_4', x) - (\partial F_1/\partial u_k')u_k' = 0, \\
& \partial F_1/\partial u_k' = 0 \qquad (k = 1, 2, 4).
\end{aligned}
$$

The equations (12A) are the analogues of the usual transversality conditions. (See Bliss, loc. cit., p. 167.)

hold. In equations (12) and (13) we have five equations for the determination of the four end values x_1, $u_h(x_1)$, $h=1, 2, 3$, of an extremal Γ. In general, therefore, the problem of simultaneous maxima is not possible for the case for which the second end parameter x_1 is required to be the same for both I_1 and I_2. Hence, it will be understood in our work that x_1 has, in general, different values for I_1 and I_2. The conditions (12) and (13) are functional in form and will be called *functional transversality conditions*.

6. Analogue of the Weierstrass necessary condition. By the aid of the expression for $\delta I_1(\Gamma)$ we can state the following theorem:

THEOREM 3. *The value of an integral I_1, taken along a two-parameter family of extremal arcs E_{01} determined by the equations $u_k=f_k(x, a_1, a_2)$, $k=1, 2, 4$, $G=0$, and the hypersurface $u_3=f_3(x)=$constant, one of whose end points, x_0, is fixed while the other, x_1, varies, has a differential*

$$dI_1 = F_1(u_1,p_1,u_2,p_2,u_3,u_3',u_4,p_4,x)dx_1$$
$$+ [\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k')][du_k(x_1) - p_k dx_1],$$

where at the point 1, the differentials dx_1 and du_k are those belonging to V_3 ($u_3=$constant) described by the end points of the extremals, while the u_i, u_3, p_i and u_3' refer to the extremal E_{01}. The functions F_1 and G have arguments (u_1, p_1, u_2, p_2, u_3, u_3', u_4, p_4, x), where the p_1, p_2 and p_4 are the directional coefficients of the extremal E_{01} for $u_3=$constant.

There is an entirely analogous theorem for the integral I_2. For I_2 the functions F_2 and G have arguments (u_1, u_1', u_2, u_2', u_3, p_3, u_4, p_4, x).

The integral of dI_1 corresponds to the Hilbert integral and possesses similar properties. In a manner analogous to the classical method of the calculus of variations it is possible to obtain the necessary conditions of Weierstrass and Legendre and to obtain sufficient conditions for relative strong and weak maxima.* To do this we first define an extremal field in the sense in which we shall use it in this chapter.

We shall say that a connected region R of the space (u_1, u_2, u_3, u_4, x) is a simply covered *extremal field* if there exists a family of extremals dependent upon three parameters such that one and only one extremal of this simply covered field passes through every point of R, and if, furthermore, the directional coefficients $du_h/dx=p_h(u_1, \cdots, u_4, x)$, $h=1, \cdots, 4$, of the tangent to the extremal, which passes through the point (u_1, u_2, u_3, u_4, x), are continuous functions, admitting continuous partial derivatives in R

* For the classical analysis see Hadamard, loc. cit., p. 364. See also Bliss, loc. cit., p. 50.

up to the second order. We shall assume that such a field exists and that it contains V_3.

It is quite evident that along an extremal arc of a field, the integral $I_1^* = \int dI_1$ has the same value as I_1, for $\delta u_i = p_i \delta x$ along an extremal, and the integrand of I_1^* thus reduces to the integrand of I_1.

To obtain an analogue of the Weierstrass condition we select a point (3) on E_{01}, the extremal which we are assuming to give the desired maximum, and through this point (3), holding $f_3(x, a_1, a_2, a_3) = u_3(x)$ constant, pass an otherwise arbitrary curve C_{12} with continuously turning tangent in R. We note that R may be partly bounded by V_3, so that when (3) is at (1) the curves C_{12} are further limited. Such a curve C_{12} will have equations $u_1 = U_1(t)$, $u_2 = U_2(t)$, $u_3 = u_3(t)$, $u_4 = U_4(t)$.

We join the fixed point 0 to a movable point 2 on C_{12} by a one-parameter family of arcs E_{02}, containing E_{01} as a member when the point 2 is in the position 1. We choose the parameter t on C_{21} increasing as 2 moves towards 1, noting that the arc length s is a possible t. If E_{01} is to give a maximum for admissible arcs in R, i.e., $I_1(E_{02}+C_{21}) \leqq I_1(E_{01})$, where C_{12} and E_{02} are obtained by putting $a_3 = 0$, it follows that $dI_1(E_{02}+C_{21}) = dI_1(E_{02}-C_{12})$ must be $\geqq 0$ for 2 sufficiently close to 1, and in particular at 1 itself, that is, $dI_1(C_{12}-E_{02}) \leqq 0$ must hold.

This differential is given by the value at the point 1 of the expression

$$
\begin{aligned}
&F_1(U_1, U_1', U_2, U_2', u_3, u_3', U_4, U_4')\delta x \\
&\qquad - F_1(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x)\delta x \\
&\qquad - [\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k')][\delta u_k - u_k' \delta x],
\end{aligned}
$$

the differentials in this expression belonging to the arc C_{12} and, therefore, satisfying the equation $\delta u_k = U_k' \delta x$. At the point 1 the coördinates of C_{12} and E_{02} are equal, so that this expression can be written as

$$
\begin{aligned}
dI_1(C_{12} - E_{02}) = {} & [F_1(u_1, U_1', u_2, U_2', u_3, u_3', u_4, U_4', x) \\
& - F_1(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x) \\
& - [U_k' - u_k'][\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u')]]\delta x.
\end{aligned}
$$

We shall call the coefficient of δx in the above expression E_1, because it is an analogue of the Weierstrass E-function.† Since the differential $dI_1(C_{12}-E_{01})$ must be negative or zero for an arbitrarily selected point 1 and an arc C through it, we have the following theorem:

† Bliss, loc. cit., p. 130.

THEOREM 4. *At every element* $(u_1, u_1', \cdots, u_4, u_4', x)$ *of an arc* E_{01} *which maximizes an integral* I_1 *when* u_3 *is not allowed to vary and which satisfies a differential equation* $G(u_1, u_1', \cdots, u_4, u_4', x)=0$, *the Weierstrass condition*

$$(14)\qquad E_1(u_1,u_1',U_1',u_2,u_2',U_2',u_3,u_3',u_4,u_4',U_4',x) \leqq 0$$

must be satisfied for every admissible set $(u_1', U_1, u_2, U_2', u_3, u_3', u_4, U_4')$, *different from* $(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x)$, *for all values of the coördinates* (u_1, u_2, u_3, u_4, x) *in the region R.*

A similar analysis applied to the integral I_2 yields the following theorem:

THEOREM 5. *At every element* $(u_1, u_1', \cdots, u_4, u_4', x)$ *of an arc* E_{01} *which maximizes an integral* I_2 *when* u_1 *and* u_2 *are not allowed to vary and which satisfies a differential equation,* $G=0$, *the condition*

$$(15)\qquad E_2(u_1,u_1',u_2,u_2',u_3,u_3',U_3',u_4',u_4',U_4',x) \leqq 0$$

must be satisfied for every admissible set $(u_1, u_1', u_2, u_2', u_3, U_3', u_4, U_4', x)$ *different from* $(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x)$, *for all values of the coördinates* (u_1, u_2, u_3, u_4, x) *in the region R.*

The conditions (11), (12), (13), (14) and (15) are necessary conditions which must be satisfied by an arc E_{01} which furnishes a solution of the problem of this paper. In the following paragraph another necessary condition will be obtained.

7. **Analogue of the Legendre necessary condition.** For brevity we will consider only the first integral I_1. If the function $F_1(u_1, U_1', u_2, U_2', u_3, u_3', u_4, U_4', x)$ be expanded by means of Taylor's formula, the following expression is obtained:

$$\begin{aligned} F_1(u_1,U_1',u_2,U_2',u_3,u_3',u_4,U_4',x) &= F_1(u_1,U_1',u_2,U_2',u_3,u_3',u_4,U_4',x) \\ &\quad + [U_k' - u_k'][\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial u_4'/\partial u_k')] \\ &\quad + \tfrac{1}{2}[U_k' - u_k'][U_h' - u_h']\partial A_{1h}/\partial u_k', \end{aligned}$$

where $A_{1h}=\partial F_1/\partial u_h' + (\partial F_1/\partial u_4')(\partial u_4'/\partial u_h')$ and h and k are umbral indices with range 1, 2. The arguments of A_{1h} are $(u_1,\ u_1'+\theta(U_1'-u_1'),\ u_2,\ u_2'+\theta(U_2'-u_2'),\ u_3,\ u_3',\ u_4,\ u_4'+\theta(U_4'-u_4'),\ x)$, where $0<\theta<1$. It should be noted that in this formula u_3 is not allowed to vary.

Since the partial derivative of G with respect to u_k determines $\partial u_4'/\partial u_k'$, the function E_1 is given by the formula

$$E_1(u_1,u_1',U_1',\cdots,u_4,u_4',U_4',x) = \tfrac{1}{2}(U_k' - u_k')(U_h' - u_h')(\partial A_{1h}/\partial u_k').$$

Let us write $T=U_1'-u_1'$, $W=U_2'-u_2'$ and $V=U_3'-u_3'$. We may then by the help of (14) state the following theorem:

THEOREM 6. *If the extremal E_{01} makes I_1 a maximum when u_3 is not allowed to vary, and at the same time makes I_2 a maximum when u_1 and u_2 are not allowed to vary, it is necessary that the quadratic differential forms*

$$T^2[\partial A_{11}/\partial u_1'] + TW[\partial A_{12}/\partial u_1' + \partial A_{11}/\partial u_2'] + W^2[\partial A_{12}/\partial u_2'], \tag{16}$$

$$V^2[\partial A_{23}/\partial u_3'] \tag{17}$$

be definite negative forms for all systems of finite values of u_1', u_2', u_3' and u_4', when the point (u_1, u_2, u_3, u_4, x) remains in the domain R.†

In (16) u_3 is not allowed to vary and in (17) u_1 and u_2 are not allowed to vary. In particular if we let U_i' approach u_i' in (16) we obtain a condition analogous to the Legendre condition.‡

8. **The analogue of the Jacobi condition.** We proceed now to determine an analogue of the Jacobi condition for the problem of simultaneous maxima. Let E_{02} and E_{03} be two extremals of a two-parameter family, $u_3 =$ constant, $u_4(x_0) = u_{40}$, through the point 0, and suppose that these extremals touch an envelope N of the family at their end points 2 and 3. Since the differential dI_1 of §6 is a total differential, u_3 being constant, the integral $I_1^* = \int dI_1$ around a closed contour C is zero. As already pointed out in §6, I_1^* along an extremal is identically equal to I_1, hence

$$I_1(E_{03}) - I_1(E_{02}) = I_1^*(N_{23}).$$

The differentials dx, du_i, $i = 1, \cdots, 4$, at a point of the envelope satisfy the equations $du_i = p_i dx$ with the slope p of the extremal tangent to N at that point. It follows then that $I_1^*(N_{23})$ is the same as $I_1(N_{23})$; hence the following theorem:

THEOREM 7. *If E_{02} and E_{03} are two members of a two-parameter family, $u_3 =$ constant, of extremals through the fixed point* 0, *and if these touch an envelope N of the family at their end points* 2 *and* 3, *then the values of the integral I_1 along the arcs E_{02}, E_{03} and N_{32} satisfy the equation $I_1(E_{03}) + I_1(N_{32}) = I_1(E_{02})$ for every position of the point* 3 *preceding* 2 *on N.*

This theorem is the analogue of the envelope theorem of the calculus of variations.§ We have also a similar theorem for the integral I_2, for u_1 and u_2 held constant.

† For definition of definite negative forms see M. Bôcher, *Introduction to Higher Algebra*, p. 150.

‡ Hadamard, loc. cit., p. 391.

§ Bliss, loc. cit., pp. 140–141.

Consider now the value of $I_1'(0)=\delta I_1(0)$ given by equation (10B) for all end values fixed. Let $\lambda_k(u_1, u_1', \cdots, u_4, u_4', x)$ and $\zeta_k(u_1, u_1', \cdots, u_4, u_4', x)$ be, respectively, the coefficients of δf_k and $\delta f_k'$ in this expression. In order to simplify the notation further let $\delta f_k=\xi_k$. We can then write (10B) as

$$I_1'(0) = \int_{x_0}^{x_1}[\lambda_k\xi_k + \zeta_k\xi_k']dx,$$

where k is umbral and ranges over 1 and 2 only. By a differentiation of this expression we obtain

$$\begin{aligned} I_1''(0) = \int_{x_0}^{x_1}[(\partial\lambda_k/\partial u_h)\xi_k\xi_h &+ (\partial\lambda_k/\partial u_h')\xi_h'\xi_k + (\partial\lambda_k/\partial u_4)\xi_k\delta u_4 \\ &+ (\partial\lambda_k/\partial u_4')\xi_k\delta u_4' + (\partial\zeta_k/\partial u_h)\xi_h\xi_k' + (\partial\zeta_k/\partial u_h')\xi_k'\xi_h' \\ &+ (\partial\zeta_k/\partial u_4)\xi_k'\delta u_4 + (\partial\zeta_k/\partial u_4')\xi_k'\delta u_4']dx, \end{aligned}$$

since ξ_3 is zero by hypothesis. The variations δu_4 and $\delta u_4'$ are to be determined by the equations of partial variation of §5. We could carry out the substitution and application of Dirichlet's formula as before, and obtain an equation analogous to Jacobi's differential equation.* As one can readily see, this equation would be functional-differential in form, and would, therefore, be extremely difficult to handle. Instead of attempting to derive Jacobi's condition rigorously by means of this equation, we shall content ourselves with using a simpler and less rigorous method.†

According to the last theorem the value of I_1 along the composite arc $E_{03}+N_{32}+E_{21}$ is always the same as its value along E_{02}. Since N_{32} is not an extremal, it can be replaced by an arc C_{34} giving I_1 a larger value, and hence $I_1(E_{01})$ cannot be a minimum.‡ As a further necessary condition we must, therefore, demand that there be no point 2 conjugate to 0 between 0 and 1 on a maximizing arc E_{01} which is an extremal, with a condition analogous to the condition $\partial^2F_1/\partial y'\partial y'\neq 0$ everywhere on it.

IV. Sufficient conditions for simultaneous maxima

9. Relative strong and weak maxima. By definition an extremal curve E_{01} furnishes a *strong relative maximum* for an integral I_1 when u_3 is not allowed to vary, if there exist a positive number ϵ such that the integral

* Bliss, loc. cit., p. 163.

† Bliss, loc. cit., p. 141.

‡ To prove this we need to know that the functional-differential equations (9) defining an extremal have a unique solution at an arbitrarily selected point and direction.

$$I_1 = \int_{x_0}^{x_1} F_1(u_1, u_1', \cdots, u_4, u_4', x)dx$$

is greater than the integral

$$I_1(w) = \int_{x_0}^{x_1} F_1[u_1 + w_1(x),\ u_1' + w_1'(x),\ \cdots,\ u_4 + w_4(x),\ u_4' + w_4'(x),\ x]dx$$

for all possible forms of the functions w_σ, $\sigma = 1, \cdots, 4$, of class (I) in the interval $x_0 \leqq x \leqq x_1$, and satisfying the conditions*

(18) $$w_\sigma(x_0) = 0\ ;\ \ |\, w_\sigma(x)\,| < \epsilon\ ;\ \ x_0 \leqq x \leqq x_1.$$

When in addition to the conditions above the functions $w_\sigma(x)$ satisfy $|\,w_\sigma'(x)\,| < \epsilon$ for $x_0 \leqq x \leqq x_1$, an extremal curve (u_1, u_2, u_3, u_4, x) furnishes a *weak relative maximum.*†

10. **Sufficient conditions for a maximum.** By means of the definition of an extremal field of §6 and the definitions of strong and weak relative maxima of §9 we are now in a position to write sufficient conditions for both strong and weak relative maxima.

THEOREM 8. *If E_{01} is an extremal arc and if the conditions* (14) *and* (15) *without the equality sign are satisfied at every element* $(u_1, u_1', \cdots, u_4, u_4', x)$ *in a neighborhood R', contained in R, of the corresponding elements of E_{01} for every admissible set* $(u_1, U_1', \cdots, u_4, U_4', x)$ *such that in* (14) *the expressions* $(U_1' - u_1')$ *and* $(U_2' - u_2')$ *are not both zero and yet* $U_3' - u_3' \equiv 0$, *and, furthermore, in* (15) $(U_3' - u_3')$ *is not zero and yet* $U_1' - u_1' \equiv 0$ *and* $U_2' - u_2' \equiv 0$, *and if, finally, there is no point* 2 *conjugate to* 0 *between* 0 *and* 1 *on E_{01}, then $I_1(E_{01})$ is a strong relative maximum when u_3 is not allowed to vary, and $I_2(E_{01})$ is a strong relative maximum when u_1 and u_2 are not allowed to vary.*

The conditions for a weak relative maximum do not require that the Weierstrass conditions be satisfied.

THEOREM 9. *If E_{01} is an extremal arc and if the Legendre conditions* (16) *and* (17) *without the equality sign are satisfied at every set of values* $(u_1, u_1', \cdots, u_4, u_4', x)$ *on this arc, and if there is no point* 2 *conjugate to* 0 *between* 0 *and* 1 *on E_{01}, then $I_1(E_{01})$ is a weak relative maximum when u_3 is not allowed to vary, and $I_2(E_{01})$ is a weak relative maximum when u_1 and u_2 are not allowed to vary.*

* If $w(x)$ is a continuous function admitting a continuous derivative in $x_0 \leqq x \leqq x_1$, we shall say that it belongs to the class (I) in the interval $(x_0 x_1)$. See E. Goursat, *Cours d'Analyse Mathématique*, vol. 3, p. 547.

† These are the classical definitions given by Goursat, loc. cit., pp. 612–613.

Although the above sufficient conditions apply strictly to the generalized Lagrange problem, by a slight modification they can be made to apply to the classical problem where only one integral I_1 is considered. Thus, allowing u_3 to vary in F_1 requires that the subscript k in (14) and (15) take on the values 1, 2, 3 instead of 1 and 2 only. The arguments of F_1 and G in all of the relations must of course be changed so that U_3' is given a proper place. In as much as this change will be obvious to the careful reader no attempt to write the corresponding conditions for the classical problem will be made here.

*The sufficiency theorems for the general problem when one end point is variable differ from those just given in that the transversality conditions must be adjoined.**

V. Integral equation treatment of the problem of Lagrange for more than one differential equation

11. Equations of variation. It is readily seen that the analysis of the preceding chapter applies to the problem of Lagrange for one differential equation. By introducing the theory of Volterra integral equations† this analysis can be modified to apply to the Lagrange problem for more than one differential relation and to the more general problem for which the differential relation is replaced by a functional relation of the type referred to in the introduction. Since the method employed in solving the problem for two differential equations is perfectly general, we need only discuss this case.

Our problem is to determine, through two fixed points 0 and 1 in the hyperspace (u_1, u_2, u_3, x), a curve E_{01} which satisfies two differential equations $G_k(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x)=0$, $k=1, 2$, and which furnishes a maximum for an integral

$$I = \int_{x_0}^{x_1} F(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x)dx.$$

We assume the G_k to be functionally independent, i.e.

$$\Delta G = \begin{vmatrix} \partial G_1/\partial u_2' & \partial G_1/\partial u_3' \\ \partial G_2/\partial u_2' & \partial G_2/\partial u_3' \end{vmatrix} \neq 0$$

in the region under consideration, and to possess continuous second-order partial derivatives with respect to $u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x$. Although

* Bliss, loc. cit., pp. 169–170.

† For the theory of Volterra integral equations see V. Volterra, *Leçons sur les Equations Intégrales.*

we have chosen both ends fixed, it is not necessary to make this assumption, as will appear presently. Let us first consider the problem stated above for both end points fixed.

Let the maximizing curve E_{01}, if such a curve exist, be the one defined by the equations

$$u_p = z_p(x) \qquad (p = 1,2),$$

$$u_3 = y(x) \qquad (x_0 \leqq x \leqq x_1)$$

and write

$$u_3 = y + \theta(x,a)\,;\ u_p = z_p + f_p(x,a) \qquad (p = 1,2),$$

where θ and f_p are functions continuous with their second derivatives with respect to x and a, and which vanish when a vanishes. This notation for the u_i, $i=1, 2, 3$, is used to indicate that u_3 is to be regarded as the function whose variation is independent. We assume the variations of u_1 and u_2 to be determined by the end values $u_1(x_0)=u_{10}$, $u_2(x_0)=u_{20}$ and the differential equations of total variation

$$(19)\quad (\partial G_k/\partial y)\delta\theta + (\partial G_k/\partial y')\delta\theta' + \sum_{p=1}^{2}[(\partial G_k/\partial z_p)\delta f_p + (\partial G_k/\partial z_p')\delta f_p'] = 0 \qquad (k = 1,2)$$

obtained by substituting the values of u_i given above in the differential equations $G_k=0$, differentiating with respect to a and then setting $a=0$. As we showed in §3 this is consistent with the assumption that 0 is a fixed end point.

If these same values of u_i be substituted in F, the integral I becomes a function of the parameter a and yields on differentiation with respect to this parameter

$$(\partial I/\partial a)\delta a = \int_{x_0}^{x_1}[(\partial F/\partial y)\delta\theta + (\partial F/\partial y')\delta\theta' + (\partial F/\partial z_r)\delta f_r + (\partial F/\partial z_r')\delta f_r']dx,$$

where r is an umbral index with range 1, 2 corresponding to p with range 1, 2.

12. **Dependent variations by the theory of Volterra integral equations.** In the classical treatment of this problem, Lagrange multipliers are introduced at this stage, but they can be advantageously avoided by integrating equation (19) with respect to x between the limits x_0 and x, where $x_0 \leqq x \leqq x_1$. If we perform this integration, replace x under the integral sign by s, and then perform an integration by parts on the terms which involve the primed variations, we obtain

$$(20)\quad (\partial G_k/\partial z_r')\delta f_r + (\partial G_k/\partial y')\delta\theta + \int_{x_0}^{x}\left[\partial G_k/\partial y - \frac{d}{ds}(\partial G_k/\partial y')\right]\delta\theta ds + \int_{x_0}^{x}\left[\partial G_k/\partial z_r - \frac{d}{ds}(\partial G_k/\partial z_r')\right]\delta f_r\, ds = 0,$$

for, since $\delta\theta$ vanishes at x_0, the δf_r must also vanish if the determinant ΔG is not zero in $x_0 \leqq x \leqq x_1$. We take r umbral as before.

The variations δf_r are then determined by the system of Volterra integral equations

$$(21)\quad \delta f_r(x) = \phi_r(x) + \int_{x_0}^{x} K_{rp}(x,s)\delta f_p(s)ds \qquad (r = 1,2),$$

where by definition

$$\phi_r(x) = A_{hr}(x)\left[(\partial G_h/\partial y')\delta\theta + \int_{x_0}^{x}\left[\partial G_h/\partial y - \frac{d}{ds}(\partial G_h/\partial y')\right]\delta\theta\, ds\right],$$

$$K_{rp}(x,s) = A_{hr}(x)\left[(\partial G_h(s)/\partial z_p) - \frac{d}{ds}(\partial G_h(s)/\partial z_p')\right],$$

where h and p are umbral indices having ranges 1, 2, and A_{hr} is the cofactor of the corresponding element of ΔG divided by $-\Delta G$.

These integral equations form a Volterra system of the second type for the determination of the δf_r, uniquely, if the kernels $K_{rp}(x, s)$ are finite and integrable in the interval $x_0 \leqq s \leqq x \leqq x_1$.*

If $\Delta G \neq 0$ on the range $x_0 \leqq s \leqq x \leqq x_1$, the $K_{rp}(x, s)$ will be finite and integrable on the range because of the continuity requirements on the G_k. The unique solution of the system is, therefore,

$$(22)\quad \delta f_r(x) = \phi_r(x) + \int_{x_0}^{x} S_{rp}(x,s)\phi_p(s)ds,$$

where p is umbral with range 1, 2 and $S_{rp}(x, s)$ is the resolvent kernel of $K_{rp}(x, s)$ defined by the equations

$$K_{rp}^{1}(x,s) = -K_{rp}(x,s),$$

$$K_{rp}^{i}(x,s) = \int_{s}^{x} K_{rh}^{e}(x,t)K_{hp}^{i-e}(t,s)dt,$$

$$S_{rp}(x,s) = \sum_{i=1}^{\infty} K_{rp}^{i}(x,s).$$

* Volterra, loc. cit., p. 71.

If we substitute in (22) the values of $\phi_r(x)$ and $\phi_p(s)$ as given by their definitions, then apply Dirichlet's formula to the iterated integral of the result, and then interchange the parameters of integration, we may write the variation of δf_r as

$$\delta f_r(x) = W_r(x)\delta\theta + \int_{x_0}^{x} V_r(x,s)\delta\theta\, ds, \tag{23}$$

where by definition

$$W_r(x) = A_{hr}(x)(\partial G_h/\partial y'),$$

$$V_r(x,s) = A_{hr}(x)\left[\partial G_h/\partial y - \frac{d}{ds}(\partial G_h/\partial y')\right] + A_{hp}(s)S_{rp}(x,s)(\partial G_h/\partial y')$$

$$+ A_{hp}(s)\left[\partial G_h/\partial y - \frac{d}{ds}(\partial G_h/\partial y')\right]\int_s^x S_{rp}(x,t)dt,$$

and where h and p are umbral indices with range 1, 2.

By differentiation with respect to x we obtain

$$\delta f_r'(x) = \frac{d}{dx}[W_r\delta\theta] + V_r(x,x)\delta\theta + \int_{x_0}^{x}[\partial V_r(x,s)/\partial x]\delta\theta\, ds. \tag{24}$$

13. Eulerian equations. A substitution of $\delta f_r(x)$ and $\delta f_r'(x)$ in the first variation of I followed by an application of Dirichlet's formula as before yields

$$(\partial I/\partial a)\delta a = \int_{x_0}^{x_1}[\partial F/\partial y + (\partial F/\partial z_r)W_r(x) + V_r(x,x)(\partial F/\partial z_r') + T(x)]\delta\theta\, dx$$

$$+ \int_{x_0}^{x_1}[(\partial F/\partial y')\delta\theta' + (\partial F/\partial z_r')d(W_r\delta\theta)/dx]dx,$$

where

$$T(x) = \int_{x}^{x_1}[(\partial F/\partial z_r)V_r(s,x) + (\partial F/\partial z_r')(\partial V_r(s,x)/\partial s)]ds.$$

An integration by parts performed on the primed terms yields

$$\delta I = \int_{x_0}^{x_1} [\partial F/\partial y + (\partial F/\partial z_r)W_r + V_r(x,x)(\partial F/\partial z_r') + T(x) - \frac{d}{dx}(\partial F/\partial y')$$

$$- W_r\frac{d}{dx}(\partial F/\partial z_r')]\delta\theta\, dx.$$

In order that δI vanish for all $\delta\theta$ it is necessary that the coefficient of $\delta\theta$ in the above integral vanish and hence

$$(25)\quad \partial F/\partial y + (\partial F/\partial z_r)W_r + V_r(x,x)\,(\partial F/\partial z_r') + T(x) - \frac{d}{dx}(\partial F/\partial y') - W_r\frac{d}{dx}(\partial F/\partial z_r') = 0,$$

where r is an umbral index with range 1, 2.

It is well to note that the function $T(x)$ is an integral involving the resolvent kernel of the system of Volterra integral equations defining the variations. If some variable other than u_3 had been chosen to be independent, a different set of conditions of the type (25) would result, but presumably the new set would be equivalent to (25).

If one or both end points were variable, the problem could still be treated by the methods of this paragraph. For this case equation (20) would contain terms in $\delta\theta(x_0)$, $\delta\theta(x_1)$, $\delta f_r(x_0)$, $\delta f_r(x_1)$. These terms could be carried all the way through the analysis and would yield transversality conditions in a new form. This interesting problem will not be attacked in the present paper.

VI. Further generalizations

14. Problem for functional relations. A special problem in which a linear integral equation replaces the first-order differential equation $G=0$ has already been considered.* We desire now to consider the more general problem of determining a curve E_{01} of the space (u_1, u_2, u_3, x) satisfying functional relations

$$(26)\quad G_k(u_1,u_1',u_2,u_2',u_3,u_3',x) = \int_{x_0}^{x} P_k(u_1,u_1',u_2,u_2',u_3,u_3',x,s)ds \qquad (k=1,2),$$

such that an integral

$$I = \int_{x_0}^{x_1} F(u_1,u_1',u_2,u_2',u_3,u_3',x)dx$$

is a maximum. We may suppose the end parameter x_0 and the corresponding end values $u_i(x_0)$, $i=1, 2, 3$, to be fixed, although this is not necessary. Let us, for the sake of brevity, also suppose x_1 and the corresponding end values of the u_i to be fixed.

If the u_i are replaced by functions satisfying the same conditions as the corresponding functions of §11, the functional equations (26) become relations involving the parameter a and yield by parametric differentiation

* *A mathematical theory of competition*, loc. cit., p. 173.

$$(\partial G_k/\partial y)\delta\theta + (\partial G_k/\partial y')\delta\theta' + (\partial G_k/\partial z_r)\delta f_r + (\partial G_k/\partial z_r')\delta f_r'$$
$$= \int_{x_0}^{x} [(\partial P_k/\partial y)\delta\theta + (\partial P_k/\partial y')\delta\theta'$$
$$+ (\partial P_k/\partial z_r)\delta f_r + (\partial P_k/\partial z_r')\delta f_r']ds \qquad (k = 1,2),$$

where r is umbral with range 1, 2.

An integration with respect to x followed by an integration by parts on the primed variations yields

$$(\partial G_k/\partial y')\delta\theta + \int_{x_0}^{x}\left[\partial G_k/\partial y - \frac{d}{ds}(\partial G_k/\partial y')\right]\delta\theta\, ds + (\partial G_k/\partial z_r')\delta f_r$$
$$+ \int_{x_0}^{x}\left[\partial G_k/\partial z_r - \frac{d}{ds}(\partial G_k/\partial z_r')\right]\delta f_r\, ds$$
$$= \int_{x_0}^{x} [(\partial P_k/\partial y')\delta\theta + (\partial P_k/\partial z_r')\delta f_r]ds$$
$$+ \int_{x_0}^{x} ds \int_{x_0}^{s}\left[\left(\partial P_k/\partial y - \frac{d}{dt}(\partial P_k/\partial y')\right)\delta\theta\right.$$
$$\left. + \left(\partial P_k/\partial z_r - \frac{d}{dt}(\partial P_k/\partial z_r')\right)\delta f_r\right]dt,$$

where for convenience in notation the parameter of integration x has been changed to s.

If we apply Dirichlet's formula to the iterated integral, we can write this expression as

$$(\partial G_k/\partial z_r')\delta f_r = -(\partial G_k/\partial y')\delta\theta - \int_{x_0}^{x}\left[\partial G_k/\partial y - \frac{d}{ds}(\partial G_k/\partial y') - \partial P_k/\partial y'\right.$$
$$\left. - (x-s)\left(\partial P_k/\partial y - \frac{d}{ds}(\partial P_k/\partial y')\right)\right]\delta\theta\, ds$$
$$- \int_{x_0}^{x}\left[\partial G_k/\partial z_r - \frac{d}{ds}(\partial G_k/\partial z_r') - \partial P_k/\partial z_r'\right.$$
$$\left. - (x-s)\left(\partial P_k/\partial z_r - \frac{d}{ds}(\partial P_k/\partial z_r')\right)\right]\delta f_r\, ds.$$

As far as the variations of δf_r and $\delta\theta$ are concerned this expression is of the same form as (21), if $\partial G_k/\partial z_r'$ is not zero on the range $x_0 \leqq s \leqq x \leqq x_1$. The analysis of the preceding section, therefore, applies from this point.

15. **Further extensions.** The extension to the case of more than one independent variable is obtained by placing a subscript on y in the above equations and regarding this subscript as an umbral index of the proper range.

The problem of simultaneous maxima for more than one differential or integral relation can be treated by this same method, since, if there are two integrals and two independent variables u_3 and u_4, equation (25), with the proper arguments for F_1, F_2 and G_k substituted, is a necessary condition that a curve E_{01} in the space (u_1, u_2, u_3, u_4, x) satisfy differential equations $G_k(u_1, u_1', \cdots, u_4, u_4', x) = 0$, $k = 1, 2$, and make an integral

$$I = \int_{x}^{x_1} F_1(u_1, u_1', \cdots, u_4, u_4', x)dx$$

a maximum when u_4 is not allowed to vary.

A discussion of the corresponding problems for variable end points should lead to analogues of the Weierstrass and Legendre necessary conditions and to sufficient conditions for strong and weak relative maxima.

The assumption that the admissible arcs have continuously turning tangents is by no means necessary. If we make assumptions on admissible arcs similar to those of Part II, we can obtain the analogues of the more general Euler equations (7) and (8), by integrating by parts the terms in $\delta\theta$ instead of those in $\delta\theta'$. The analogues of the Weierstrass-Erdmann corner conditions* follow readily from the Euler equations in the form (7) and (8).

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* Bliss, loc. cit., p. 143, and O. Bolza, *Vorlesungen über Variationsrechnung*, 1909, p. 366.

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ON JACOBI'S ARITHMETICAL THEOREMS CONCERNING THE SIMULTANEOUS REPRESENTATION OF NUMBERS BY TWO DIFFERENT QUADRATIC FORMS*

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PART I

The infinite series and products afforded by the theory of elliptic functions lead in a natural and easy way to the discovery of many peculiar arithmetical theorems which, at first sight, do not seem to be easily obtainable by purely arithmetical methods. But deeper insight into arithmetical properties of elliptic series and products shows that the use of them may be superseded by certain arithmetical relations of a very general nature. Liouville was the first to call attention to this fact, but he never attempted to give a complete and systematic account of his ideas, and for this reason they did not attract the attention they deserve. The author of this paper, by his personal investigations concerning this subject, has been led to the conclusion that all the results previously obtained by means of elliptic functions may be as well established by purely arithmetical methods of extremely elementary nature, and he published his investigations, so far as it was possible under the circumstances, in a series of papers.† It is his intention to show in this paper how the arithmetical theorems obtained by Jacobi in his memoir *Ueber diejenigen unendlichen Reihen, deren Exponenten zugleich in zwei verschiedenen quadratischen Formen enthalten sind*‡ can be easily derived by very simple arithmetical considerations. The paper is divided into two parts, according to two different methods of treatment for questions of this kind. The method developed in the first part is based on Liouville's ideas, but,

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† J. Uspensky, *On arithmetical theorems given by Stieltjes* (in Russian), Bulletin of the Mathematical Society in Kharkov, vol. 13 (1912); *On the representation of numbers by sums of squares* (in Russian), ibid., vol. 13 (1912); *On certain arithmetic theorems* (in Russian), ibid., vol. 13 (1913); *On the possibility of representation of primes by certain quadratic forms* (in Russian), Bulletin of the Mathematical Society in Kazan, 1915; *On the representation of numbers by the quadratic forms with 4 and 6 variables* (in Russian), Bulletin of the Mathematical Society in Kharkov, vol. 16 (1916); *Sur les relations entre les nombres des classes des formes quadratiques binaires et positives*, Parts 1, 2, 3, 4, 5, Bulletin of the Academy of Sciences of the U. S. S. R., 1925–26; *Note sur le nombre de représentations des nombres par une somme d'un nombre pair de carrés*, ibid., 1925.

‡ Jacobi's Gesammelte Werke, vol. 3, p. 220.

though simple and elegant, it is not the best adapted to this subject. Such a method, meeting all the requirements of simplicity and efficacy, will be developed in the second part.

1. **The fundamental identity and its consequences.** Instead of using various properties and transformations of elliptic series and products, we shall take for our starting point a single very general formula which, in a certain sense, contains the very essence of the arithmetical properties of elliptic functions. Let $F(x, y, z)$ represent a function defined for integral values of the variables x, y, z and satisfying the conditions

$$(1)\quad F(-x, y, z) = -F(x, y, z), \qquad F(x, -y, -z) = F(x, y, z),$$
$$F(0, y, z) = 0.$$

For any such function and for any positive integer n, the following fundamental relation holds:

$$(2)\quad 2\sum F(\delta - 2i, d + i, 2d + 2i - \delta) - \sum F(d + \delta, i, d - \delta) = T,$$

where both sums extend over all the solutions of the indeterminate equation

$$(3)\qquad n = i^2 + d\delta$$

in integers i, d, δ, the two last being supposed positive, and $T=0$ when n is not a perfect square,

$$T = \sum_{j=1}^{2s-1}[2F(2s-j, s, 2s-j) - F(2s, s-j, 2s-2j)] \text{ when } n = s^2,\ s > 0.$$

In order to avoid lengthy explanations, it is useful to introduce the symbolic notation $\{A\}$ for a quantity related to an integer n, having value 0 except when n is a perfect square: $n=s^2$, $s>0$, in which case the above symbol represents the number A. With this notation adopted, the fundamental equation (2) may be written as follows:

$$(2\text{ bis})\quad \begin{aligned} &2\sum_{(a)}[F(\delta - 2i, d + i, 2d + 2i - \delta) - F(d + \delta, i, d - \delta)] \\ &= \left\{\sum_{j=1}^{2s-1}\left[2F(2s-j, s, 2s-j) - F(2s, s-j, 2s-2j)\right]\right\},\ n = s^2,\ s > 0, \\ &(a)\qquad n = i^2 + d\delta.\text{*} \end{aligned}$$

The proof of this important equation is very easy, and the reader can find it for instance in the first part of our memoir *Sur les relations entre les nombres des classes des formes quadratiques binaires et positives.*†

* In this formula (a) $n=i^2+d\delta$ indicates the range of summation for $\sum_{(a)}$. A similar notation is used throughout.

† Loc. cit.

From the fundamental equation (2), by submitting the highly arbitrary function $F(x, y, z)$ to certain restrictions, many other important relations may be derived. In this paper, however, we confine ourselves only to such of these relations as are absolutely necessary for our purpose. Denoting by $f(x)$ an arbitrary *odd* function of a single variable x, we can first take, in the fundamental equation (2),

$$F(x, y, z) = 0 \text{ for even } x, \quad F(x, y, z) = f(x) \text{ for odd } x,$$

which, after a simple discussion, leads to the following relation:

$$\sum_{(a)} f(2d + \delta) = \sum_{(b)} f(\delta - 2i) - \left\{ \sum_{(c)} f(j) \right\}, \tag{4}$$

(a) $n = i^2 + 2d\delta$, δ odd; (b) $n = i^2 + d\delta$, δ odd;

(c) $j = 1, 3, 5, \cdots, 2s - 1$, $n = s^2$, $s > 0$.

Again, assuming in the same fundamental equation

$$F(x, y, z) = 0 \text{ for even } x,$$
$$F(x, y, z) = (-1)^{(x+z)/2+y} f(y) \text{ for odd } x,$$

we arrive at the very useful formula

$$\sum_{(a)} (-1)^i f(d + i) = \left\{ (-1)^{s-1} s f(s) \right\}, \quad n = s^2, \; s > 0, \tag{5}$$

(a) $n = i^2 + d\delta$, δ odd.

Suppose now n divisible by 4, so that $n = 4m$. By taking first for $F(x, y, z)$ the function defined as follows:

$$F(x, y, z) = 0, \text{ whenever } x \text{ is odd},$$
$$F(x, y, z) = 0, \text{ whenever } y \text{ is even},$$
$$F(x, y, z) = \phi(x/4, y), \text{ otherwise},$$

where $\phi(x, y)$ denotes an arbitrary function *odd* with respect to x and *even* with respect to y, we have after a simple discussion and with a slightly changed notation

$$\sum_{(a)} \phi[(d + \delta)/4, \lambda] = 2 \sum_{(b)} \phi(d + i, \delta - 2i) + \left\{ 2 \sum_{(c)} \phi(s, j) \right\}, \tag{6}$$

(a) $4m = \lambda^2 + d\delta$, λ odd; (b) $m = i^2 + d\delta$, δ odd;

(c) $j = 1, 3, 5, \cdots, 2s - 1$, $m = s^2$, $s > 0$.

Taking again

$$F(x, y, z) = 0, \text{ whenever } x \text{ is odd},$$
$$F(x, y, z) = 0, \text{ whenever } y \text{ is even},$$
$$F(x, y, z) = (-1)^{(y-z/2)/2} \phi(x/4, y), \text{ otherwise},$$

we get the equation

$$(7)\quad \sum_{(a)}(-1)^{(\lambda-1)/2+(d-\delta-2)/4}\phi[(d+\delta)/4,\ \lambda] = 2\sum_{(b)}(-1)^{d}\phi(d+i,\ \delta-2i) + \left\{2\sum_{(c)}\phi(s,\ j)\right\},$$

$(a)\quad 4m=\lambda^2+d\delta,\ \lambda$ odd; $\qquad (b)\quad m=i^2+d\delta,\ \delta$ odd;

$(c)\quad j=1, 3, 5, \cdots, 2s-1,\quad m=s^2,\ s>0,$

the left member of which is obviously equal to 0. Subtracting now (7) from (6) we have finally

$$(8)\qquad \sum_{(a)}\phi[(d+\delta)/4,\ \lambda] = 4\sum_{(b)}\phi(d+i,\ \delta-2i),$$

$(a)\quad 4m=\lambda^2+d\delta,\ \lambda$ odd; $\qquad (b)\quad m=i^2+d\delta,\ d$ and δ odd.

The last equation we need may be obtained as follows. First we suppose n divisible by 3, so that $n=3m$, and then define $F(x, y, z)$ as follows:

$F(x, y, z) = 0$, whenever $y+z$ is not divisible by 3,

$F(x, y, z) = 0$, whenever y is divisible by 3,

$F(x, y, z) = f(x/3)$ otherwise,

$f(x)$ being an arbitrary *odd* function of x. It is obvious that this definition is consistent with the fundamental conditions (1). The resulting equation with a slightly modified notation may be written as follows:

$$(9)\quad \sum_{(a)}f[(\Delta+\Delta')/3] = 2\sum_{(b)}f(\delta-2i) + \{4sf(2s)\},\ m=3s^2,\ s>0,$$

$(a)\quad 3m=h^2+\Delta\Delta',\ h+\Delta-\Delta'\equiv 0 \pmod 3;\qquad (b)\quad m=3i^2+d\delta,\ d\not\equiv 0 \pmod 3,$

and admits of a further simplification. As h is supposed to be non-divisible by 3 in the equation $3m=h^2+\Delta\Delta'$, it is easy to see that, whenever $h+\Delta-\Delta'$ is divisible by 3, $-h+\Delta-\Delta'$ is *not* divisible by 3, and, whenever $h+\Delta-\Delta'$ is not divisible by 3, $-h+\Delta-\Delta'$ is divisible by 3, whence it follows that

$$\sum_{(a)}f[(\Delta+\Delta')/3] = \sum_{(b)}f[(\Delta+\Delta')/3],$$

$(a)\quad h+\Delta-\Delta'\equiv 0,\ h\not\equiv 0 \pmod 3;\qquad (b)\quad h+\Delta-\Delta'\not\equiv 0,\ h\not\equiv 0 \pmod 3,$

or

$$2\sum_{(a)}f[(\Delta+\Delta')/3] = \sum_{(b)}f[(\Delta+\Delta')/3],$$

$(a)\quad h+\Delta-\Delta'\equiv 0,\ h\not\equiv 0 \pmod 3;\qquad (b)\quad m=h^2+\Delta\Delta',\ h\not\equiv 0 \pmod 3.$

The left hand member of (9) being thus simplified, we finally get the important relation

$$(10)\qquad \sum_{(a)} f[(\Delta + \Delta')/3] = 4\sum_{(b)} f(\delta - 2i) + \{8sf(2s)\},\ m = 3s^2,\ s > 0,$$

(a) $3m = h^2 + \Delta\Delta',\ h \not\equiv 0 \pmod 3$; (b) $3m = 3i^2 + d\delta\ \ d \not\equiv 0 \pmod 3$.

2. **Number of representations by certain quadratic forms.** In the following discussion we need to use the well known results concerning the number of representations by quadratic forms x^2+y^2, x^2+2y^2 and x^2+3y^2. The discussion would be considerably abbreviated should we simply recall these results, but, from a methodical point of view, it is of interest to derive all the auxiliary propositions from one and the same source.

Beginning with the quadratic form x^2+y^2, let us consider the equation $n=x^2+y^2$, and denote by $N(n)$ the number of its solutions. The following two equations are obvious:

$$N(4n) = N(n),$$
$$N(n) = 0, \text{ when } n \equiv 3 \pmod 4.$$

Furthermore, denoting by m any odd number, let us consider the equation

$$2m = x^2 + y^2 + z^2.$$

We do not assume that this equation is necessarily solvable, but, if it is, among the numbers x, y, z there are two odd and one even. The number of solutions with an *odd* x is, therefore, twice as great as the number of solutions with an *even* x, which leads to the recurrent relation

$$\sum_{(a)} N(2m - x^2) = 2\sum_{(b)} N(2m - 4x^2),$$

(a) $x = \pm 1, \pm 3, \pm 5, \cdots$; (b) $x = 0, \pm 1, \pm 2, \cdots$,

both series being continued so far as the arguments remain positive. The same relation may be also written in the form

$$(11)\qquad \sum_{(a)} (-1)^x N(2m - x^2) + \sum_{(a)} N(2m - 4x^2) = 0,$$

(a) $x = 0, \pm 1, \pm 2, \cdots$.

If there are no representations of $2m$ by the sum of three squares (which is really impossible), both the sums in (11) are equal to 0, and (11) continues to hold.

Considering the equation $m=x^2+y^2+z^2$, and supposing first $m\equiv 1$

(mod 4), we easily see that the number of solutions with an *even* x is twice as great as the number of solutions with an *odd* x, so that

$$\sum_{(a)} N(m-4x^2) = 2\sum_{(b)} N(m-x^2),$$

(a) $x = 0, \pm 1, \pm 2, \cdots;$ (b) $x = \pm 1, \pm 3, \pm 5, \cdots,$

or, what is the same thing,

$$\sum_{(a)} N(m-x^2) = \sum_{(b)} (-1)^x N(m-x^2), \tag{12}$$

(a) $x = \pm 1, \pm 3, \pm 5, \cdots;$ (b) $x = 0, \pm 1, \pm 2, \cdots,$

both series being continued as long as the arguments remain non-negative. In the case $m \equiv 3 \pmod 4$ we have

$$-\sum_{(a)} N(m-x^2) = \sum_{(b)} (-1)^x N(m-x^2), \tag{13}$$

(a) $x = \pm 1, \pm 3, \pm 5, \cdots;$ (b) $x = 0, \pm 1, \pm 2, \cdots,$

because $N(m-4x^2)=0$ for every x. The equations (12) and (13) can be condensed into one,

$$(-1)^{(m-1)/2}\sum_{(a)} N(m-x^2) = \sum_{(b)} (-1)^x N(m-x^2), \tag{14}$$

(a) $x = \pm 1, \pm 3, \pm 5, \cdots;$ (b) $x = 0, \pm 1, \pm 2, \cdots,$

which thus holds for every odd m. Now it is possible to find a priori a numerical function satisfying the same equations (11) and (14). For this purpose we take

$$f(x) = (-1)^{(x-1)/2}$$

in the relation (4); the resulting equation

$$\sum_{(a)} (-1)^{d+(\delta-1)/2} = \sum_{(b)} (-1)^{i+(\delta-1)/2} - \left\{\tfrac{1}{2}(1+(-1)^{s-1})\right\},\ n = s^2,\ s > 0,$$

(a) $n = i^2 + 2d\delta$, δ odd; (b) $n = i^2 + d\delta$, δ odd,

introducing the numerical function $\rho(n)$ defined by

$$\rho(n) = \sum (-1)^{(\delta-1)/2},\ n = d\delta,\ \delta \text{ odd};\ \rho(0) = \tfrac{1}{4},$$

leads, after a simple discussion, to the following relations:

$$\sum_{(a)} (-1)^x \rho(2m-x^2) + \sum_{(b)} \rho(2m-4x^2) = 0, \tag{15}$$

(a) $x = 0, \pm 1, \pm 2, \cdots,$ (b) $x = 0, \pm 1, \pm 2, \cdots;$

$$(-1)^{(m-1)/2}\sum_{(a)} \rho(m-x^2) = \sum_{(b)} (-1)^x \rho(m-x^2),$$

(a) $x = \pm 1, \pm 3, \pm 5, \cdots;$ (b) $x = 0, \pm 1, \pm 2, \cdots,$

perfectly analogous to (11) and (14). Putting $w(n) = N(n) - 4\rho(n)$, we have therefore

$$\sum_{(a)}(-1)^x w(2m - x^2) + \sum_{(b)}{}' w(2m - 4x^2) = 0,$$

(a) $x = 0, \pm 1, \pm 2, \cdots;$ (b) $x = 0, \pm 1, \pm 2, \cdots;$

$$\sum_{(a)}(-1)^x w(m - x^2) = (-1)^{(m-1)/2}\sum_{(b)} w(m - x^2),$$

(a) $x = 0, \pm 1, \pm 2, \cdots;$ (b) $x = \pm 1, \pm 3, \pm 5, \cdots,$

or

$$w(2m) + \sum_{(a)}(-1)^x w(2m - x^2) + \sum_{(b)} w(2m - 4x^2) = 0, \tag{16}$$

(a) $x = 1, 2, 3, \cdots;$ (b) $x = 1, 2, 3, \cdots;$

$$w(m) + 2\sum_{(a)}(-1)^x w(m - x^2) - 2(-1)^{(m-1)/2}\sum_{(b)} w(m - x^2) = 0, \tag{17}$$

(a) $x = 1, 2, 3, \cdots;$ (b) $x = 1, 3, 5, \cdots.$

Moreover, it follows from the definition of $\rho(n)$ that $\rho(4n) = \rho(n)$, so that for every n

$$w(4n) = w(n). \tag{18}$$

Now $w(0) = 0$ and we can establish that $w(n) = 0$ for every n by means of mathematical induction. For, supposing the equation $w(n) = 0$ established for all $n < N$, we can prove that $w(N) = 0$. First, if N is divisible by 4, we have, by (18), $w(N) = w(N/4) = 0$. If $N = 2m$, where m is supposed to be odd, it follows from (16) that $w(N) = 0$. Finally, for an odd N the same conclusion follows from (17). The equation $w(n) = 0$ being thus established, we have reached at the same time the well known result

$$N(n) = 4\rho(n)$$

by means of extremely simple considerations.

The number of representations by the form $x^2 + 2y^2$ may be obtained in a very similar way. Denoting again by $N(n)$ the number of solutions of the equation $n = x^2 + 2y^2$, we have first the obvious equation $N(2n) = N(n)$, and the following equation can be very easily verified:

$$N(n) = 0 \quad \text{when} \quad n \equiv 5 \text{ or } \equiv 7 \pmod 8.$$

Consider now the equation

$$m = 4x^2 + y^2 + 2z^2,$$

where m is supposed to be *odd*; the number of its solutions is given either by the sum

$$\sum N(m - 4x^2) \qquad (x = 0, \pm 1, \pm 2, \cdots),$$

or by the sum

$$\sum N(m - x^2) \qquad (x = \pm 1, \pm 3, \pm 5, \cdots),$$

so that we have

$$(19) \qquad \sum_{(a)} N(m - 4x^2) = \sum_{(b)} N(m - x^2),$$

$$(a) \quad x = 0, \pm 1, \pm 2, \cdots; \qquad (b) \quad x = \pm 1, \pm 3, \pm 5, \cdots.$$

On the other hand, the function $\phi(n)$ defined by

$$\phi(n) = \sum (-1)^{(\delta-1)/2+(\delta^2-1)/8} = \sum \left(\frac{-2}{\delta}\right), \; n = d\delta, \; \delta \text{ odd},$$

$$\phi(0) = \tfrac{1}{2},$$

satisfies the equation of the same form. To show this we take $f(x) = \sin(\pi x/4)$ in the relation (4). Noticing that for any odd number x

$$\sin \frac{\pi x}{4} = \frac{1}{2^{1/2}}(-1)^{(x-1)/2+(x^2-1)/8},$$

it is easy to get the following two equations: for $m \equiv 1 \pmod 4$,

$$(-1)^{(m-1)/4} \sum_{(a)} \phi(m - x^2) = \sum_{(b)} (-1)^x \phi(m - 4x^2),$$

$$(a) \quad x = \pm 1, \pm 3, \pm 5, \cdots \qquad (b) \quad x = 0, \pm 1, \pm 2, \cdots,$$

and for $m \equiv 3 \pmod 4$,

$$(-1)^{(m-3)/4} \sum_{(a)} \phi(m - x^2) = \sum_{(b)} (-1)^x \phi(m - 4x^2),$$

$$(a) \quad x = \pm 1, \pm 3, \pm 5, \cdots; \qquad (b) \quad x = 0, \pm 1, \pm 2, \cdots.$$

But as $\phi(n) = 0$ when $n \equiv 5$ or $7 \pmod 8$, it is easy to see that

$$[(-1)^{(m-1)/4+x} - 1]\phi(m - 4x^2) = 0, \text{ when } m \equiv 1 \pmod 4,$$

$$[(-1)^{(m-3)/4+x} - 1]\phi(m - 4x^2) = 0, \text{ when } m \equiv 3 \pmod 4,$$

whence it follows that the two preceding equations may be combined into one:

$$\sum_{(a)} \phi(m - x^2) = \sum_{(b)} \phi(m - 4x^2),$$

$$(a) \quad x = \pm 1, \pm 3, \pm 5, \cdots; \qquad (b) \quad x = 0, \pm 1, \pm 2, \cdots,$$

which is perfectly analogous to (19). Putting, therefore, $w(n)=N(n)-2\phi(n)$, we have

$$w(m)+2\sum_{(a)}w(m-4x^2)=2\sum_{(b)}w(m-x^2), \tag{20}$$

(a) $x=1,2,3,\cdots$; (b) $x=1,3,5,\cdots$;

and obviously $w(2n)=w(n)$. Now it is easy to show that, for every n, $w(n)=0$. For, suppose this already verified for all $n<N$. If N be an even number, then $w(N)=w(N/2)=0$. But if N be an odd number, the same conclusion follows from (20). As $w(0)=0$, we have $w(n)=0$ for every n, that is, $N(n)=2\phi(n)$. It remains to determine by means of analogous considerations the number of representations by the form x^2+3y^2. Denoting again by $N(n)$ the number of solutions of the equation $n=x^2+3y^2$, the following properties of $N(n)$ are almost evident. First,

$$N(3n)=N(n),$$

$$N(n)=0 \text{ when } n\equiv 2 \pmod 3,$$

and furthermore, if m denotes an odd number,

$$N(2^\alpha m)=0 \text{ for an } \textit{odd } \alpha,$$

$$N(2^\alpha m)=N(4m) \text{ for an } \textit{even } \alpha.$$

That the equation $N(4m)=3N(m)$ holds true for every odd m, is not so evident, but still can be easily proved. It implies that the number of solutions of the equation $4m=x^2+3y^2$ in *odd* numbers x, y is twice as great as the number of all solutions of the equation $m=x^2+3y^2$. Consider now the equation

$$4n=\lambda^2+\mu^2+3\nu^2,$$

where λ and ν are supposed to be odd. The number of its solutions, on the one hand, is given by

$$\sum N(4n-\lambda^2) \qquad (\lambda=\pm 1,\pm 3,\pm 5,\cdots);$$

on the other hand it follows from the preceding remark that the same number is given by the sum $2\sum N(n-\kappa^2)$ extended over all integers κ such that $n-\kappa^2$ is positive and *odd*. Thus we get the important equation

$$\sum N(4n-\lambda^2)=2\sum N(n-\kappa^2),\quad n-\kappa^2 \text{ odd}. \tag{21}$$

We shall show now that the function $\chi(n)$ defined by

$$\chi(n)=\sum\left(\frac{-3}{\delta}\right),\quad n=d\delta,\ \delta \text{ odd},$$

satisfies exactly the same equation for every n non-divisible by 3. To this end we take in the relation (8)

$$\phi(x, y) = \sin\frac{8\pi x}{3}\cos\frac{2\pi y}{3},$$

which leads to the following equation:

$$(22)\qquad \sum_{(a)} \sin\frac{2\pi d}{3}\cos\frac{2\pi\delta}{3}\cos\frac{2\pi\lambda}{3} = 2\sum_{(b)}\sin\frac{2\pi d}{3}\cos\frac{2\pi\delta}{3}\cos\frac{2\pi i}{3},$$

(a) $4n = \lambda^2 + d\delta$, λ odd; (b) $n = i^2 + d\delta$, d and δ odd.

In order to exhibit it in the simplest possible form we notice first that

$$\cos\frac{2\pi\alpha}{3} = 1 \text{ if } \alpha \equiv 0 \pmod 3,$$

$$\cos\frac{2\pi\alpha}{3} = -\frac{1}{2} \text{ if } \alpha \not\equiv 0 \pmod 3,$$

and that for every odd number x

$$\sin\frac{2\pi x}{3} = \frac{3^{1/2}}{2}\left(\frac{-3}{x}\right).$$

Profiting by these remarks we find for $n \equiv 1 \pmod 3$

$$\cos\frac{2\pi\delta}{3}\cos\frac{2\pi\lambda}{3} = -\frac{1}{2},\quad \cos\frac{2\pi\delta}{3}\cos\frac{2\pi i}{3} = -\frac{1}{2},$$

so that equation (22) in this case may be written as follows:

$$\sum\chi(4n - \lambda^2) = 2\sum\chi(n - i^2),\ n - i^2 \text{ odd}.$$

For $n \equiv 2 \pmod 3$ we have

$$\cos\frac{2\pi\delta}{3}\cos\frac{2\pi\lambda}{3} = -\frac{1}{2} \text{ or } +\frac{1}{4},\ \text{if } \lambda \equiv 0 \text{ or } \equiv \pm 1 \pmod 3,$$

$$\cos\frac{2\pi\delta}{3}\cos\frac{2\pi i}{3} = -\frac{1}{2} \text{ or } +\frac{1}{4},\ \text{if } i \equiv 0 \text{ or } \equiv \pm 1 \pmod 3,$$

and equation (22) becomes

$$-\sum_{(a)}\chi(4n - \lambda^2) + \tfrac{1}{2}\sum_{(b)}\chi(4n - \lambda^2) = -2\sum_{(c)}\chi(n - i^2) + \sum_{(d)}\chi(n - i^2),$$

(a) $\lambda \equiv 0$; (b) $\lambda \equiv \pm 1$; (c) $i \equiv 0$; (d) $i \equiv \pm 1 \pmod 3$.

But for $n \equiv 2 \pmod 3$ it is easy to show that $\chi(n) = 0$, whence it follows that the preceding equation may be also written in the form

$$\sum \chi(4n - \lambda^2) = 2 \sum \chi(n - i^2), \quad n - i^2 \text{ odd}.$$

Thus, for every n non-divisible by 3 we have

$$\sum \chi(4n - \lambda^2) = 2 \sum \chi(n - i^2), \quad n - i^2 \text{ odd},$$

and, putting $w(m) = N(m) - 2\chi(m)$ for an odd m, we reach the analogous equation for $w(n)$,

$$\sum w(4n - \lambda^2) = 2 \sum w(n - \kappa^2), \quad n - \kappa^2 \text{ odd}, \tag{23}$$

provided $n \equiv \pm 1 \pmod 3$. For small odd values of m we find $w(m) = 0$ and to prove this in general we shall apply the process of reasoning by induction.

First we have $w(m) = 0$ whenever $m \equiv 2 \pmod 3$. Now we assume as already proved that $w(m) = 0$ for all $m \equiv 3 \pmod 4$ which are $< 12k + 7$ and for all $m \equiv 1 \pmod 4$ which are smaller than $4k + 1$. Let us take in the equation (23) $n = 3k + 1$ and $n = 3k + 2$ respectively; in virtue of this supposition all the terms in the second member disappear, and we get

$$w(12k + 3) + w(12k - 5) + \cdots = 0,$$
$$w(12k + 7) + w(12k - 1) + \cdots = 0.$$

By supposition,

$$w(12k - 5) = 0, \quad w(12k - 21) = 0, \cdots,$$
$$w(12k - 1) = 0, \quad w(12k - 17) = 0, \cdots,$$

so that

$$w(12k + 3) = w(4k + 1) = 0,$$
$$w(12k + 7) = 0.$$

Taking again in equation (23) $n = 3k + 4$, we get

$$w(12k + 15) + w(12k + 7) + \cdots = 0,$$

whence

$$w(12k + 15) = w(4k + 5) = 0.$$

Moreover, $w(12k + 11) = 0$, and consequently we have $w(m) = 0$ for all $m \equiv 3 \pmod 4$ which are $< 12(k+1) + 7$ and for all $m \equiv 1 \pmod 4$ which are $< 4(k+1) + 1$, that is, the induction is complete and the equation $w(m) = 0$ established. Thus we have $N(m) = 2\chi(m)$ for any odd m, while $N(4m) = 6\chi(m)$, and in general

$$N(2^\alpha m) = 6\chi(m), \text{ when } \alpha \text{ is even}, > 0,$$
$$N(2^\alpha m) = 0, \text{ when } \alpha \text{ is odd}.$$

3. **Gauss-Jacobi Theorem.** Taking in equation (5) $n=8p+3$ and

$$f(x) = \sin\frac{\pi x}{4},$$

we get

$$\sum_{(a)}(-1)^i \cos\frac{\pi i}{4}\sin\frac{\pi d}{4} = 0,$$

(a) $\qquad n = i^2 + d\delta,\ \delta$ odd,

which is equivalent to

$$\sum_{(a)}(-1)^h\phi(n-16h^2) = \sum_{(b)}(-1)^{(i^2-1)/8}\rho\left(\frac{n-i^2}{2}\right),$$

(a) $\quad h = 0, \pm 1, \pm 2, \cdots;$ $\qquad$ (b) $\quad i = \pm 1, \pm 3, \pm 5, \cdots,$

$\rho(n)$ and $\phi(n)$ being taken with the same meaning as above. As $\phi(n-16h^2)$ and $\rho[(n-i^2)/2]$ represent respectively the numbers of solutions of the equations

$$n - 16h^2 = k^2 + 2l^2, \quad l \text{ odd} > 0,$$
$$n - i^2 = 2l^2 + 8j^2, \quad l \text{ odd} > 0,$$

it is obvious that the preceding equation can be also written in the form

$$\sum_{(a)}(-1)^h = \sum_{(b)}(-1)^{(i^2-1)/8}, \tag{24}$$

(a) $\quad n = k^2 + 2l^2 + 16h^2;$ $\qquad$ (b) $\quad n = i^2 + 2l^2 + 8j^2.$

Now let us consider two numerical functions $G(m)$ and $g(m)$ defined for every $m \equiv 1 \pmod 8$ by

$$G(m) = \sum(-1)^y, \quad m = x^2 + 16y^2,$$
$$g(m) = \sum(-1)^{(x^2-1)/8}, \quad m = x^2 + 8y^2,$$

the sums being extended over all the solutions of the corresponding equations. Introducing these functions, the relation (24) can be written as follows:

$$\sum G(8p+3-2l^2) = \sum g(8p+3-2l^2) \qquad (l = 1, 2, 3, \cdots);$$

whence it follows necessarily that

$$G(8p+1) = g(8p+1),$$

that is, for every $m \equiv 1 \pmod 8$ we have

$$\sum_{(a)}(-1)^y = \sum_{(b)}(-1)^{(x^2-1)/8},$$

(a) $\quad m = x^2 + 16y^2;$ $\qquad$ (b) $\quad m = x^2 + 8y^2,$

and this constitutes one of the theorems given by Jacobi. In the particular case when m is a prime number, both of the equations

$$m = a^2 + 16b^2, \quad m = c^2 + 8d^2$$

possess only one solution in positive integers, and it follows from the preceding equality that whenever b is even, c is of the form $8l \pm 1$, and whenever b is odd, c is of the form $8l \pm 3$. This particular theorem was first found by Gauss and published in his first memoir on biquadratic residues.

4. **Another theorem by Jacobi. Preliminary results.** As the second example of the use of the same kind of reasoning we choose another more complicated theorem given by Jacobi. Certain preliminary propositions are, however, to be established first. Taking $f(x) = \sin(3\pi x/2)$ in (10), the resulting equation may be presented as follows:

$$\sum_{(a)}(-1)^{d+(\delta-1)/2} = 2\sum_{(b)}(-1)^{i-1}\sin\frac{\pi\delta}{2},$$

(a) $\quad 3n = h^2 + 2d\delta,\ h \not\equiv 0 \pmod 3),\ \delta$ odd; $\quad$ (b) $\quad n = 3i^2 + d\delta,\ d \not\equiv 0 \pmod 3),$

or else

$$2\sum(-1)^{i-1}\rho(n - 3i^2) = \sum(-1)^{(n+h^2)/2}\rho(3n - h^2), \tag{25}$$

the last sum being extended over all h which are not divisible by 3 and of the same parity as n. Suppose now that n is not divisible by 3. In this case for h divisible by 3,

$$\rho(3n - h^2) = 0,$$

because the number $3n - h^2$ contains the prime number 3 only to the first power, and therefore the summation in the right hand member of (25) may be extended over all h of the same parity as n. Taking into account the arithmetical meaning of the function $\rho(n)$ the equation (25) leads to the relations

$$\sum_{(a)}(-1)^i = (1/2)(-1)^{(n-2)/2}N(3n = h^2 + 2k^2 + 2l^2),$$

$$\sum_{(a)}(-1)^i = (1/2)(-1)^{(n-1)/2}N(3n = h^2 + 2k^2 + 2l^2),$$

(a) $$n = 3i^2 + j^2 + k^2,$$

for even and odd values of $n \equiv \pm 1 \pmod 3$ respectively. Supposing further that $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 3$, the two preceding equations can be combined into one, namely

$$(-1)^{n-1}\sum_{(a)}(-1)^k = \tfrac{1}{4}N(3n = x^2 + 2y^2 + 2z^2),$$

(a) $$n = 9i^2 + j^2 + 3k^2.$$

But it is easy to show that

$$\tfrac{1}{4}N(3n = x^2 + 2y^2 + 2z^2) = N(n = x^2 + 2y^2 + 6z^2),$$

and so finally we get the first preliminary result,

$$N(n = x^2 + 2y^2 + 6z^2) = (-1)^{n-1}\sum(-1)^k, \tag{26}$$

for every $n \equiv 1 \pmod{12}$ or $\equiv 4 \pmod{12}$.

Now we put $f(x) = \sin(2\pi x/3)$ in the equation (5). Denoting by σ_1 and σ_2 the two sums

$$\sigma_1 = \sum(-1)^i \sin\frac{2\pi d}{3}, \quad n = 9i^2 + d\delta, \quad \delta \text{ odd},$$

$$\sigma_2 = -\frac{1}{2}\sum(-1)^i \sin\frac{2\pi d}{3}, \; n = i^2 + d\delta, \; \delta \text{ odd}, \; i \not\equiv 0 \pmod 3,$$

extended over all corresponding representations of n, we have

$$\sigma_1 + \sigma_2 = \frac{3^{1/2}}{2}\left\{(-1)^{s-1}\left(\frac{s}{3}\right)s\right\}, \quad n = s^2, \quad s > 0.$$

By means of the known expression for the number of representations by the form x^2+3y^2 it is easy to express σ_1 and σ_2 as follows. Denote by P, Q, R, S, T the numbers of the solutions of the equations

$$n = 9i^2 + j^2 + 3k^2 \quad \text{with } j + k \text{ odd},$$

$$n = 9i^2 + j^2 + 3k^2 \quad \text{with } j + k \text{ even},$$

$$n = i^2 + j^2 + 3k^2 \quad \text{with } j + k \text{ odd and } i \not\equiv 0 \pmod 3,$$

$$n = i^2 + j^2 + 3k^2 \quad \text{with } j + k \text{ even and } i \not\equiv 0 \pmod 3,$$

$$n = i^2 + 2j^2 + 6k^2\,;$$

then

$$\sigma_1 = \frac{3^{1/2}}{4}(-1)^{n-1}\left(P - \frac{1}{3}Q\right),$$

$$\sigma_2 = \frac{3^{1/2}}{8}(-1)^{n-1}\left(-R + \frac{1}{3}S - \frac{1}{3}T\right),$$

and consequently

$$P - \frac{1}{2}R - \frac{1}{3}Q - \frac{1}{6}S - \frac{1}{6}T = 2\left\{\left(\frac{s}{3}\right)s\right\}, \quad n = s^2, \quad s > 0. \tag{27}$$

By (26),

$$T = (-1)^{n-1}\sum(-1)^k, \quad n = 9i^2+j^2+3k^2,$$

or

$$T = \sum_{(a)}(-1)^{i+n-1} + \sum_{(b)}(-1)^{i+n-1} - \sum_{(c)}(-1)^{i+n-1},$$

(a) $i\equiv n, j\equiv n$; (b) $i\equiv n-1, j\equiv n$; (c) $i\equiv n, j\equiv n-1 \pmod 2$

all the sums being extended over solutions of the equation $n=9i^2+j^2+3k^2$, subject to the limitations indicated below each sign of the sum. For P, Q, R and S we find easily analogous expressions

$$P = \sum_{(a)}(-1)^{i+n-1}, \quad Q = -\sum_{(b)}(-1)^{i+n-1} - \sum_{(c)}(-1)^{i+n-1},$$

(a) $i\equiv n-1, j\equiv n$; (b) $i\equiv n, j\equiv n-1$; (c) $i\equiv n, j\equiv n \pmod 2$;

$$R = -\sum_{(a)}(-1)^{i+n-1}, \quad S = \sum_{(b)}(-1)^{i+n-1} - \sum_{(c)}(-1)^{i+n-1},$$

(a) $i\equiv n, j\equiv n-1$; (b) $i\equiv n-1, j\equiv n$; (c) $i\equiv n, j\equiv n \pmod 2$.

Now, substituting all these expressions into (27) we get

$$\sum_{(a)}(-1)^{i+n-1} + \sum_{(b)}(-1)^{i+n-1} = 2\left\{\left(\frac{s}{3}\right)s\right\}, \quad n = s^2, s > 0,$$

(a) $i\equiv n-1, j\equiv n$; (b) $i\equiv n, j\equiv n-1 \pmod 2$,

or finally

$$\sum_{(a)}(-1)^i = (-1)^{n-1}2\left\{\left(\frac{s}{3}\right)s\right\}, \quad n = s^2, \; s > 0, \tag{28}$$

(a) $n = 9i^2+j^2+3k^2$; $i+j$ odd.

If, supposing $n\equiv 1 \pmod 4$, we put in (5) $f(x)=\sin(\pi x/2)$, the resulting equation may be easily presented as follows:

$$\sum_{(a)}(-1)^h = 2\left\{(-1)^{(s-1)/2}s\right\}, \quad n = s^2, \; s > 0, \tag{29}$$

(a) $n = i^2+4h^2+4k^2$.

5. **Proof of Jacobi's theorem.** Denoting by m any odd number we shall consider two sums

$$\sigma_1 = \sum\left(\frac{x}{3}\right)x,$$

$$\sigma_2 = \sum(-1)^{(x-1)/2+(y-1)/2}\left(\frac{x}{3}\right)y,$$

extended over all solutions of the equation $4m=x^2+3y^2$ in odd numbers x and y. All these solutions satisfying an additional condition $x\equiv y \pmod 4$

can be derived from the solutions of the equation $m=\xi^2+3\eta^2$ by the substitution $x=\xi+3\eta$, $y=\xi-\eta$, whence it follows that

$$\sigma_1 = 2\sum\left(\frac{\xi}{3}\right)(\xi+3\eta) = 2\sum\left(\frac{\xi}{3}\right)\xi,$$

$$\sigma_2 = 2\sum\left(\frac{\xi}{3}\right)(\xi-\eta) = 2\sum\left(\frac{\xi}{3}\right)\xi,$$

or

$$\sigma_1 = \sigma_2.$$

This being established we introduce the sum

$$S = \sum(-1)^{(x-1)/2+z}\left(\frac{x}{3}\right),$$

extended over all the solutions of the equation $4n=x^2+3(y^2+4z^2+4t^2)$, where x is supposed odd, while n represents any given number. Applying (29) we readily get

$$S = \sum(-1)^{(x-1)/2+(y-1)/2}\left(\frac{x}{3}\right)y,\ 4n = x^2+3y^2,\ x \text{ odd},$$

that is, 0 for n even and σ_2 for n odd. Since $\sigma_2=\sigma_1$, we have in all cases

$$S = \sum\left(\frac{x}{3}\right)x,\ 4n = x^2+3y^2,\ x \text{ odd},$$

the right hand member being naturally 0 for an even n. On the other hand, we can express S as follows:

$$S = \sum\psi(4n-3y^2) \qquad (y = \pm 1, \pm 3, \pm 5, \cdots),$$

introducing the function $\psi(m)$ defined by

$$\psi(m) = \sum(-1)^{(x-1)/2+y}\left(\frac{x}{3}\right);\ m = x^2+12y^2+12z^2$$

for all $m\equiv 1 \pmod 4$. Comparing the two expressions for S we get

$$\sum_{(a)}\psi(4n-3y^2) = \sum_{(b)}\left(\frac{x}{3}\right)x,$$

(a) $\quad y = \pm 1, \pm 3, \pm 5, \cdots$ $\qquad$ (b) $\quad 4n = x^2+3y^2$, y odd,

whence it follows that

$$\psi(n) = 2\left\{\left(\frac{s}{3}\right)s\right\},\ n = s^2,\ s>0,$$

for every $n\equiv 1 \pmod 4$, or

$$(30)\qquad \sum_{(a)}(-1)^{(x-1)/2+y}\left(\frac{x}{3}\right) = 2\left\{\left(\frac{s}{3}\right)s\right\},\quad n = s^2,\ s > 0,$$

$$(a)\qquad n = x^2 + 12y^2 + 12z^2.$$

The right hand member here being the same as in (28), we have

$$\sum_{(a)}(-1)^{y} = 2\sum_{(b)}(-1)^{x+y},$$

$$(a)\quad n = x^2 + 9y^2 + 12z^2;\qquad (b)\quad n = (6x+1)^2 + 12y^2 + 12z^2,$$

and as this equation is true for every $n\equiv 1 \pmod{12}$, it follows necessarily that

$$(-1)^{(n-1)/4}\sum_{(a)}(-1)^{y} = 2\sum_{(b)}(-1)^{x},$$

$$(a)\quad n = x^2 + 9y^2;\qquad (b)\quad n = (6x+1)^2 + 12y^2,$$

for every $n\equiv 1 \pmod{12}$. This is Jacobi's theorem which we had to prove.

6. **Stephen Smith's theorem.** An interesting theorem of the same kind, but involving an indefinite form, has been given by Stephen Smith in his *Report on the theory of numbers*. In order to derive this theorem by our elementary methods we need to establish a certain general relation by means of a certain device which may be useful on many occasions. Returning to the fundamental equation (2) we put $4n$ instead of n and choose for $F(x, y, z)$ either

$$F(x, y, z) = 0 \text{ whenever } x \text{ is odd or } y \text{ even},$$
$$F(x, y, z) = f(x/4, (y+z)/2) \text{ otherwise},$$

or

$$F(x, y, z) = 0 \text{ whenever } x \text{ is odd or } y \text{ even},$$
$$F(x, y, z) = f(x/4, (y-z)/2) \text{ otherwise},$$

where $f(x, y)$ is an arbitrary function *odd* with respect to x and *even* with respect to y. This gives

$$\sum_{(a)} f\left(\frac{d+\delta}{4}, \frac{\lambda+\frac{1}{2}(d-\delta)}{2}\right) = 2\sum_{(b)} f(d+i, \delta-d-2i) + \left\{2\sum_{(c)} f(s, j)\right\},$$

$$(a)\quad 4n = \lambda^2 + d\delta,\ \lambda \text{ odd};\qquad (b)\quad n = i^2 + d\delta,\ \delta \text{ odd};$$

$$(c)\quad j = 1, 3, 5, \cdots, 2s-1,\ n = s^2,\ s > 0;$$

$$\sum_{(a)} f\left(\frac{d+\delta}{4}, \frac{\lambda+\frac{1}{2}(\delta-d)}{2}\right) = 2\sum_{(b)} f(d+i, d) + \left\{2\sum_{(c)} f(s, 0)\right\},$$

$$(a)\quad 4n = \lambda^2 + d\delta,\ \lambda \text{ odd};\qquad (b)\quad n = i^2 + d\delta,\ \delta \text{ odd}$$

$$(c)\quad = 1, 3, 5, \cdots, 2s-1,\ n = s^2,\ s > 0,$$

and as the left members are obviously equal, we get

$$\sum_{(a)} f(d+i,\ \delta-d-2i) = \sum_{(b)} f(d+i,\ d) + \{sf(s,\ 0)\} - \{\ \sum f(s,\ j)\},$$

(a) $\quad n = i^2 + d\delta,\ \delta$ odd; $\qquad$ (b) $\quad n = i^2 + d\delta,\ \delta$ odd.

Now we take n as an odd number and suppose that $f(x, y)=0$ whenever y is odd. In this case the preceding equation becomes

$$\sum_{(a)} f(d+2h,\ \delta-d-4h) = \sum_{(b)} (d+i,\ d) + \{sf(s,\ 0)\},$$

(a) $\quad n = 4h^2 + d\delta;$ $\qquad$ (b) $\quad n = i^2 + d\delta,\ i$ and δ odd.

We specialize this equation further by taking

$$f(x,\ y) = (-1)^{(x-1)/2}F(y/2),$$

where $F(y)$ is an arbitrary *even* function; after evident reductions we obtain

$$\sum_{(a)} (-1)^{(d-1)/2+h} F\left(2h + \frac{d-\delta}{2}\right) = \{(-1)^{(s-1)/2} sF(0)\}, \tag{31}$$

(a) $\quad n = 4h^2 + d\delta,$

for *every odd* number n and *even* function $F(y)$. It is remarkable, however, that this equation holds true for an absolutely arbitrary function $F(y)$ *if we suppose* $n\equiv 1 \pmod 4$. Let $F(y)$ be an odd function and $F(0)=0$; n being $\equiv 1 \pmod 4$, we evidently have

$$\sum (-1)^{(d-1)/2+h} F\left(2h + \frac{d-\delta}{2}\right) = \sum (-1)^{(\delta-1)/2+h} F\left(2h + \frac{\delta-d}{2}\right)$$
$$= -\sum (-1)^{(d-1)/2+h} F\left(2h + \frac{d-\delta}{2}\right),$$

whence

$$\sum_{(a)} (-1)^{(d-1)/2+h} F\left(2h + \frac{d-\delta}{2}\right) = 0,$$

(a) $\quad n = 4h^2 + d\delta.$

Now let $F(y)$ be an arbitrary function. The equation (31) being satisfied for

$$F_1(y) = \tfrac{1}{2}(F(y) + F(-y)),$$
$$F_2(y) = \tfrac{1}{2}(F(y) - F(-y)),$$

it will be satisfied for their sum

$$F_1(y) + F_2(y) = F(y).$$

Let k denote any number satisfying the condition $4k^2 < n$, where we suppose

$n\equiv 1 \pmod 4$. Taking $n-4k^2$ instead of n in (31) and putting $\Phi(x-2k, 2k)$, where $\Phi(x, y)$ is an arbitrary function, instead of $F(x)$, we get

$$\sum_{(a)}(-1)^{(d-1)/2+k}\Phi\left(\frac{d-\delta}{2}+2h-2k,\ 2k\right) = \left\{(-1)^{(s-1)/2}s\Phi(-2k,\ 2k)\right\},$$

$$(a) \qquad n-4k^2 = 4h^2+d\delta. \qquad\qquad n-4k^2 = s_1^2 s > 0,$$

Here we give to k all possible values consistent with the condition $4k^2<n$ and add all the resulting equations; these operations performed, we arrive at the equation

$$\sum_{(a)}(-1)^{(d-1)/2+k}\Phi\left(\frac{d-\delta}{2}+2h-2k,\ 2k\right) = \sum_{(b)}(-1)^{(s-1)/2}s\Phi(-2k,\ 2k),$$

$$(a) \quad n = 4h^2+k^2+d\delta; \qquad\qquad (b) \quad n = s^2+4k^2,\ s>0,$$

which by the substitution $y=h+k$, $z=h-k$ can be transformed into

$$\sum_{(a)}(-1)^{(d-1)/2+(y+z)/2}\Phi\left(\frac{d-\delta}{2}+2z,\ y-z\right) = \sum_{(b)}(-1)^{(s-1)/2}s\Phi(-2k,\ 2k),$$

$$(a) \quad n = 2y^2+2z^2+d\delta,\ d\delta\equiv 1 \pmod 4; \qquad\qquad (b) \quad n = s^2+4k^2,\ s>0.$$

Now, let us define an arbitrary function $\Phi(x, y)$ as follows:

$$\Phi(x,\ y) = 0 \quad \text{when } x \text{ is different from } 0,$$
$$\Phi(0,\ y) = f(y),$$

where $f(y)$ is an arbitrary function of a single variable. After a simple discussion we arrive at the formula

$$\sum_{(a)}(-1)^{(x-1)/2+(y+z)/2}f(y+z) = \left\{(-1)^{(s-1)/2}sf(0)\right\},\ n = s^2,\ s>0,$$

$$(a) \qquad n = x^2+2y^2-2z^2,\ x\pm 2z>0,$$

where the sum in the left member extends over all representations of n by the indefinite form $x^2+2y^2-2z^2$, the variables being limited by the conditions $x-2z>0$, $x+2z>0$. Putting $f(x)=(-1)^{x/2}F(x)$, we finally get

$$(32) \qquad \sum_{(a)}(-1)^{(x-1)/2}F(y+z) = \left\{(-1)^{(s-1)/2}sF(0)\right\},\ n = s^2,\ s>0,$$

$$(a) \qquad n = x^2+2y^2-2z^2,\ x\pm 2z>0,$$

for $n\equiv 1 \pmod 4$ and an absolutely arbitrary function $F(x)$. Taking here $F(x)=\cos(\pi x/2)$, we have for $n\equiv 1 \pmod 8$

$$\sum_{(a)}(-1)^{(x-1)/2+y+z} = \left\{(-1)^{(s-1)/2}s\right\},\ n = s^2,\ s>0,$$

$$(a) \qquad n = x^2+8y^2-8z^2,\ x\pm 4z>0,$$

which is equivalent to

$$(33)\qquad \sum_{(a)}(-1)^y g(n-8y^2) = \{(-1)^{(s-1)/2}s\},\quad n = s^2,\ s > 0,$$

$$(a)\qquad y = 0, \pm 1, \pm 2, \cdots,$$

where $g(n)$ represents the function defined by

$$g(n) = \sum(-1)^{(x-1)/2+z},$$

the sum being extended over all the representations of n by the indefinite form x^2-8z^2, the variables being limited by the conditions $x+4z>0$, $x-4z>0$.

On the other hand, we have, by (29),

$$\sum_{(a)}(-1)^{y+z} = 2\{(-1)^{(s-1)/2}s\},\quad n = s^2,\ s > 0,$$

$$(a)\qquad n = x^2 + 8y^2 + 8z^2,$$

that is,

$$(34)\qquad \sum_{(a)}(-1)^y G(n-8y^2) = \{(-1)^{(s-1)/2}s\},\quad n = s^2,\ s > 0,$$

$$(a)\qquad y = 0, \pm 1, \pm 2, \cdots,$$

where $G(n)$ is defined by

$$G(n) = \sum(-1)^z,\quad n = x^2 + 8z^2,\ x > 0.$$

As the left members of (33) and (34) are equal, we have

$$\sum(-1)^y g(n-8y^2) = \sum(-1)^y G(n-8y^2)$$

for every $n\equiv 1 \pmod 8$, whence it follows necessarily that $g(n)=G(n)$ for $n\equiv 1 \pmod 8$, or

$$\sum_{(a)}(-1)^{(x-1)/2+z} = \sum_{(b)}(-1)^z,$$

$$(a)\qquad n = x^2 - 8z^2,\ x \pm 4z > 0;\qquad\qquad (b)\qquad n = x^2 + 8z^2,\ x > 0,$$

which constitutes the theorem given by Stephen Smith.

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ON RELATIVE CONTENT AND GREEN'S LEMMA*

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It has been shown† that if the line integral $\int_C x dy$ exists over a simple closed plane curve C, then the content of K, the interior of C, exists equal to that integral. This result may be thought of as the special case of Green's lemma

$$(1) \qquad \iint_K P_x(xy)dxdy = \int_C P(xy)dy$$

in which $P(xy)=x$; and it is to be noted that here C does not need to be rectifiable.

In the present paper a definition of relative content is given which makes it possible to prove that if P and P_x are subject to certain conditions, the content of K, relative to a certain non-additive function of rectangles derived from P, exists equal to the double integral on the left of (1) and also equal to the line integral on the right of (1) whenever that integral exists. This result includes as a special case the form of Green's lemma for rectifiable C obtained by Gross,‡ except that in our result P_x is deliberately restricted to be properly Riemann integrable instead of summable. In the last section sufficient conditions for the existence of the line integral are given which yield Green's lemma for an important case in which C does not need to be rectifiable.

1. **Definitions and elementary theorems.** Let $\mathfrak{P}$ denote a class of partitions Π of the rectangle R_0: $a \leqq x \leqq b$, $c \leqq y \leqq d$, such that (1) each partition Π is formed by dividing R_0 into vertical and horizontal strips; and (2) the (greatest) lower bound of the norms of the partitions Π of $\mathfrak{P}$ is zero; here by the norm of a partition Π of $\mathfrak{P}$ is meant the (least) upper bound of the lengths of the diagonals of the rectangles of which Π consists.

Moreover let $f(R)$ be a function (not necessarily single-valued) defined for every rectangle R: $x' \leqq x \leqq x''$, $y' \leqq y \leqq y''$ lying in R_0. Also, if K_1 and K_2 are any two sets in R_0, let $\epsilon(K_1, K_2)=1$ if K_1 and K_2 have at least one point

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† H. L. Smith, these Transactions, vol. 27, p. 498.

‡ Wm. Gross, Monatshefte für Mathematik und Physik, vol. 26, p. 70. See also Van Vleck, Annals of Mathematics, (2), vol. 22, p. 226; Bray, Annals of Mathematics, (2), vol. 26, p. 278.

in common, and let $\epsilon(K_1, K_2)=0$ if K_1 and K_2 have no point in common. Finally let $\Delta\Pi$ denote any one of the rectangles of which Π consists and let $N\Pi$ denote the norm of Π.

Then if K is a set in R_0 and

$$\lim_{N\Pi=0} \sum_{\Delta\Pi} f(\Delta\Pi)\epsilon(K, \Delta\Pi)$$

exists, it is called the outer content of K relative to f. In the above the summation is naturally over all $\Delta\Pi$. If

$$\lim_{N\Pi=0} \sum_{\Delta\Pi} f(\Delta\Pi)[1 - \epsilon(R_0 - K, \Delta\Pi)]$$

exists, it is called the inner content of K relative to f. If both the outer and the inner contents of K relative to f exist and are equal, their common value is called the content of K relative to f.

The outer content of K relative to f exists *absolutely* if it not only exists relative to f but also relative to $|f|$. Similar definitions are given to the absolute existence of the inner content and of the content itself.

A set K is *squarable* relative to f if its content exists and equals zero; it is absolutely squarable relative to f if its content exists absolutely relative to f and is zero.

We mention the following obvious theorem:

THEOREM 1. *If a set K is absolutely squarable every subset of K is absolutely squarable.*

We now prove

THEOREM 2. *If the boundary of a set K, entirely interior to R_0, is absolutely squarable relative to f, then the content of K exists if either the inner content or the outer content exists relative to f.*

For

$$\sum_{\Delta\Pi} f(\Delta\Pi)\epsilon(K, \Delta\Pi) = \sum_{\Delta\Pi} f(\Delta\Pi)[1 - \epsilon(R_0 - K, \Delta\Pi)] + T(\Pi),$$

where

$$T(\Pi) = \sum_{\Delta\Pi} f(\Delta\Pi)\epsilon(K, \Delta\Pi)\epsilon(R_0 - K, \Delta\Pi).$$

But

$$|T(\Pi)| \leqq \sum_{\Delta\Pi} |f(\Delta\Pi)| \epsilon(K_b, \Delta\Pi),$$

where K_b denotes the boundary of K. Hence $\lim_{N\Pi=0} T(\Pi)=0$, from which the theorem follows.

THEOREM 3. *If K_1 and K_2 are both interior to R_0 and have no points in common and if each has inner content (f) and one of their boundaries is squarable absolutely (f), then the inner content (f) of K_1+K_2 exists equal to the sum of the inner contents (f) of K_1 and K_2.*

For suppose K_{1b}, the boundary of K_1, is squarable absolutely (f). Then, if we set $K=K_1+K_2$,

$$\sum_{\Delta\Pi} f(\Delta\Pi)[1-\epsilon(R_0-K,\Delta\Pi)] = \sum_{\Delta\Pi} f(\Delta\Pi)[1-\epsilon(R_0-K_1,\Delta\Pi)] + \sum_{\Delta\Pi} f(\Delta\Pi)[(1-\epsilon(R_0-K_2,\Delta\Pi)] + T(\Pi),$$

where

$$T(\Pi) = \sum_{\Delta\Pi} f(\Delta\Pi)\epsilon(K_1,\Delta\Pi)\epsilon(K_2,\Delta\Pi)[1-\epsilon(R_0-K,\Delta\Pi)].$$

But

$$|T(\Pi)| \leqq \sum_{\Delta\Pi} |f(\Delta\Pi)|\epsilon(K_{1b},\Delta\Pi).$$

Hence $\lim_{N\Pi=0} T(\Pi)=0$, from which the theorem follows.

THEOREM 4. *If K_1 and K_2 are interior to R_0 and have no points in common and if K_1 and K_2 each have outer content (f) and one of their boundaries is squarable absolutely (f), then the outer content (f) of K_1+K_2 exists equal to the sum of the outer contents (f) of K_1 and K_2.*

For suppose K_{1b}, the boundary of K_1, is absolutely squarable (f). Then

$$\sum_{\Delta\Pi} f(\Delta\Pi)\epsilon(K_1+K_2,\Delta\Pi) = \sum_{\Delta\Pi} f(\Delta\Pi)\epsilon(K_1,\Delta\Pi) + \sum_{\Delta\Pi} f(\Delta\Pi)\epsilon(K_2,\Delta\Pi) - T(\Pi),$$

where

$$T(\Pi) = \sum_{\Delta\Pi} f(\Delta\Pi)\epsilon(K_1,\Delta\Pi)\epsilon(K_2,\Delta\Pi).$$

But

$$|T(\Pi)| \leqq \sum_{\Delta\Pi} |f(\Delta\Pi)|\epsilon(K_{1b},\Delta\Pi).$$

Hence $\lim_{N\Pi=0} T(\Pi)=0$; from which the theorem follows.

Theorems 2, 3 and 4 now give

THEOREM 5. *If K_1 and K_2 are interior to R_0 and have no points in common and if K_1 and K_2 each have content (f) and one of their boundaries is absolutely squarable (f), then the content of K_1+K_2 exists (f) and equals the sum of the contents (f) of K_1 and K_2.*

We shall need the following special case of Theorem 5.

THEOREM 6. *If K is the interior of a simple closed curve C which is interior to R_0 and K has inner content (f) and C is absolutely squarable (f), then K and $C+K$ each have content (f) and their contents are equal.*

2. **On the existence of relative content.** Let $P(xy)$ and $Q(xy)$ be defined on R_0. Then let two (multiply-valued) functions $P^{(x)}(R)$ and $Q^{(y)}(R)$ be defined as follows:

$$P^{(x)}(R) = P(x''y) - P(x'y), \qquad y' \leqq y \leqq y'' ;$$
$$Q^{(y)}(R) = Q(xy'') - Q(xy'), \qquad x'' \leqq x \leqq x' ;$$

where R is the rectangle $x' \leqq x \leqq x''$, $y' \leqq y \leqq y''$, which is assumed to be in R_0. In this section we shall be interested in content relative to $P^{(x)}Q^{(y)}$ in the special case where $Q(xy) = y$.

THEOREM 7. *If P_x, the first partial derivative of P with respect to x, exists on K, the interior of a simple closed squarable curve C in R_0, and if P_x is bounded and integrable on K, then the inner content of K relative to $P^{(x)}y^{(y)}$ exists absolutely and equals $\iint_K P_x\, dxdy$.*

To prove this, note that by the mean value theorem

$$\sum_{\Delta\Pi} P^{(x)}(\Delta\Pi)y^{(y)}(\Delta\Pi)[1 - \epsilon(R_0 - K, \Delta\Pi)]$$
$$= \sum_{\Delta\Pi} P_x(\zeta^{\Delta\Pi})\Delta\Pi[1 - \epsilon(R_0 - K, \Delta\Pi)],$$

where $\zeta^{\Delta\Pi}$ is a point (xy) in $\Delta\Pi$. But

$$\sum_{\Delta\Pi} P_x(\zeta^{\Delta\Pi})\Delta\Pi[1 - \epsilon(R_0 - K, \Delta\Pi)] = \sum_{\Delta\Pi} P_x(\zeta^{\Delta\Pi})(K\cdot\Delta\Pi)\epsilon(K, \Delta\Pi) - T(\Pi),$$

where $K\cdot\Delta\Pi$ denotes the set of points common to K and $\Delta\Pi$ and where

$$T(\Pi) = \sum_{\Delta\Pi} P_x(\zeta^{\Delta\Pi})(K\cdot\Delta\Pi)\epsilon(R_0 - K, \Delta\Pi)\epsilon(K, \Delta\Pi) ;$$

here $\zeta^{\Delta\Pi}$ has already been defined if $\epsilon(R_0 - K, \Delta\Pi) = 0$ and is defined as any point (xy) in $K\cdot\Delta\Pi$ otherwise. But then

$$\lim_{N\Pi=0} \sum_{\Delta\Pi} P_x(\zeta^{\Delta\Pi})(K\cdot\Delta\Pi)\epsilon(K\cdot\Delta\Pi) = \iint_K P_x\, dxdy$$

and since

$$|T(\Pi)| \leqq N \sum_{\Delta\Pi}(\Delta\Pi)\epsilon(C, \Delta\Pi),$$

where N is the least upper bound of $|P_x|$ on K, it follows that

$$\lim_{N\Pi} T(\Pi) = 0.$$

This proves the theorem.

THEOREM 8. *If P_x exists and is bounded and integrable on R_0 and if C is a simple closed squarable curve interior to R_0, then C is absolutely squarable* $(P^{(x)}y^{(y)})$.

For

$$\sum_{\Delta\Pi} \left| P^{(x)}(\Delta\Pi)y^{(y)}(\Delta\Pi) \right| \epsilon(C, \Delta\Pi) = \sum_{\Delta\Pi} \left| P_x(\zeta^{\Delta\Pi}) \right| (\Delta\Pi)\epsilon(C, \Delta\Pi)$$

$$\leqq N \cdot \sum_{\Delta\Pi}(\Delta\Pi)\epsilon(C, \Delta\Pi).$$

Hence

$$\lim_{N\Pi} \sum_{\Delta\Pi} \left| P^{(x)}(\Delta\Pi)y^{(y)}(\Delta\Pi) \right| \epsilon(C, \Delta\Pi) = 0,$$

which is the theorem.

THEOREM 9. *If K is a region bounded by a simple closed squarable curve C and P_x exists and is bounded and integrable on R_0, then K and $K+C$ each has content equal to $\iint_K P_x\, dxdy$ relative to $P^{(x)}y^{(y)}$.*

This follows from Theorems 6, 7 and 8.

It has now become necessary to make an additional assumption concerning $\mathfrak{P}$. We say a partition Π of R_0 is of type (A) if* $x^{(x)}(\Delta\Pi)$ is constant for all $\Delta\Pi$ of Π and if $y^{(y)}(\Delta\Pi) \leqq x^{(x)}(\Delta\Pi)$ for every $\Delta\Pi$ of Π. A partition Π of R_0 is of type (B) if it can be obtained from a partition of type (A) by subdividing some or all of the cells of that partition into at most three parts each by means of vertical lines. We assume throughout the remainder of the paper that $\mathfrak{P}$ consists of all partitions of R_0 of type (A) or type (B).

We are now in a position to prove

LEMMA 1. *If C is a simple closed rectifiable curve interior to R_0, then*

$$\sum_{\Delta\Pi} y^{(y)}(\Delta\Pi)\epsilon(C, \Delta\Pi) \leqq 6 \cdot 2^{1/2}C,$$

for every Π with norm sufficiently small, where C denotes the length of C.

To prove this we note that if r is less than one-half the diameter of C,

$$C_{(r)} \leqq 2rC, \dagger$$

where $C_{(r)}$ denotes the outer content of the set of all points distant by not more than r from C. But if also $r = 2^{2/1}x^{(x)}(\Delta\Pi)$, and Π is of type (A),

---

* Naturally we are here considering only partitions of R which satisfy condition (1) of §1.

† This follows from a similar inequality for a simple arc stated by Gross, Monatshefte, vol. 29, p. 177. The proof given by Gross is incomplete; a correct proof is to be found in the author's Chicago dissertation.

$$\sum_{\Delta\Pi}(\Delta\Pi)\epsilon(C, \Delta\Pi) \leqq C_{(r)} \leqq 2rC = 2\cdot 2^{1/2}x^{(x)}(\Delta\Pi)C.$$

Hence since $\Delta\Pi = x^{(x)}(\Delta\Pi)y^{(y)}(\Delta\Pi)$ and $x^{(x)}(\Delta\Pi)$ is constant,

$$\sum_{\Delta\Pi} y^{(y)}(\Delta\Pi)\epsilon(C, \Delta\Pi) \leqq 2\cdot 2^{1/2}C,$$

which proves the result for type (A); from this the result easily follows for type (B).

THEOREM 10. *If P is continuous in x uniformly as to (xy) on R_0 and C is a simple closed rectifiable curve interior to R_0, then C is absolutely squarable $(P^{(x)}y^{(y)})$.*

For

$$\sum_{\Delta\Pi} | P^{(x)}(\Delta\Pi)y^{(y)}(\Delta\Pi) | \epsilon(C, \Delta\Pi) \leqq a(\Pi)\sum_{\Delta\Pi} y^{(y)}(\Delta\Pi)\epsilon(C, \Delta\Pi) \leqq a(\Pi)6\cdot 2^{1/2}C,$$

where $a(\Pi)$ is the largest value of $|P^{(x)}(\Delta\Pi)|$ for all $\Delta\Pi$ of Π. But on account of the uniform continuity,

$$\lim_{N\Pi} a(\Pi) = 0,$$

from which the theorem follows.

3. **On the x-linear extension of P. Its uniform continuity.** So far we have been considering the function $P(xy)$ as defined on the entire rectangle R_0. We now suppose that $P(xy)$ is defined on a closed set S interior to R_0 and show how to extend its definition to the entire plane. To this end let (x_0y_0) be a point not in S. Let $\underline{x}_0$ be the lower bound of all x' such that for $x' \leqq x \leqq x_0$ the point (xy_0) is not in S. Since S is closed it is clear that if $\underline{x}_0$ is finite the point $(\underline{x}_0y_0)$ is in S. Similarly let $\bar{x}_0$ be the upper bound of all x'' such that for $x_0 \leqq x \leqq x''$ the point (xy_0) is not in S; if $\bar{x}_0$ is finite, the point $(\bar{x}_0y_0)$ is in S. We now define $P(x_0y_0)$ as follows:

$$P(x_0y_0) = P(\underline{x}_0y_0) + (x_0 - \underline{x}_0)P(\underline{x}_0\bar{x}_0, y_0),$$

if $\underline{x}_0$, $\bar{x}_0$ are both finite, where

$$P(\underline{x}_0\bar{x}_0, y_0) = [P(\bar{x}_0y_0) - P(\underline{x}_0y_0)]/[\bar{x}_0 - \underline{x}_0].$$

We also make the following definitions: $P(x_0y_0) = P(\bar{x}_0y_0)$ if $\underline{x}_0$ only is infinite; $P(x_0y_0) = P(\underline{x}_0y_0)$ if $\bar{x}_0$ only is infinite; $P(x_0y_0) = 0$ if both $\underline{x}_0$ and $\bar{x}_0$ are infinite. We call the function whose definition has been thus extended the x-linear extension of P.

THEOREM 11. *If P is defined on and interior to a simple closed curve C, is continuous on C and has a bounded first partial derivative P_x on K, the interior of C, then the x-linear extension of P is continuous as to x uniformly as to (xy) on R_0.*

To prove this we note that since P is continuous on C it is uniformly continuous there, that is, there is a system $(d_e' \mid e)$ such that

$$| P(x_1y_1) - P(x_2y_2) | \leqq e/3$$

for (x_1y_1), (x_2y_2) on C and $[(x_1-x_2)^2+(y_1-y_2)^2]^{1/2} \leqq d_e'$.

We say two points $(x_0'y_0)$, $(x_0''y_0)$ are of the first kind if they are both on C; of the second kind if (xy_0) is inside C for every x between x_0' and x_0''; of the third kind if (xy_0) is outside C for every x between x_0' and x_0''.

Consider first a pair $(x_0'y_0)$, $(x_0''y_0)$ of the first kind. It follows from the above that for such a pair

$$| P(x_0'y_0) - P(x_0''y_0) | \leqq e/3$$

if $|x_0' - x_0''| \leqq d_e'$.

Next consider a pair of the second kind. Here by the mean value theorem

$$| P(x_0'y_0) - P(x_0''y_0) | \leqq | P_x(x'''y_0) | \, | x_0' - x_0'' | \leqq N | x_0' - x_0'' | \leqq e/3,$$

if $|x_0' - x_0''| \leqq d_e''$, where d_e'' is the smaller of d_e' and $e/(3N)$, N being the least upper bound on P_x on K, and x''' is between x_0' and x_0''.

Consider next a pair of the third kind. In this case there is a pair of the first kind and also of the third kind* $(x_0^{\mathrm{I}}y_0)$, $(x_0^{\mathrm{II}}y_0)$ such that $x_0^{\mathrm{I}} \leqq x_0'$, $x_0'' \leqq x_0^{\mathrm{II}}$. But then by definition of x-linear extension

$$| P(x_0'y_0) - P(x_0''y_0) | = |P(x_0^{\mathrm{I}}y_0) - P(x_0^{\mathrm{II}}y_0)| \, |x_0' - x_0''| / |x_0^{\mathrm{I}} - x_0^{\mathrm{II}}| \leqq e/3$$

for $|x_0' - x_0''| \leqq d_e$, where d_e is the smaller of d_e'' and $ed_e'/(6M)$, M being the least upper bound of P on C.

We now consider an arbitrary pair of points $(x_0'y_0)$, $(x_0''y_0)$ in R_0 such that $|x_0' - x_0''| \leqq d_e$. The interval $(x_0'x_0'')$ can be broken up into at most three sub-intervals each of which with y_0 gives rise to a pair of points either of the first or second or third kind. Hence

$$| P(x_0'y_0) - P(x_0''y_0) | \leqq e/3 + e/3 + e/3 = e$$

for $|x_0' - x_0''| \leqq d_e$, which proves the desired theorem.

Theorems 6, 7, 10, 11 now give

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* This is true unless $P(x_0'y_0) = P(x_0''y_0)$, in which case the result is obvious.

THEOREM 12. *If P is defined on a simple closed rectifiable curve C and its interior K, is continuous on C and possesses a bounded integrable first partial derivative P_x on K, then K and $K+C$ each have content equal to $\iint P_x dxdy$ relative to $P_1^{(x)}y^{(y)}$ and C is absolutely squarable relative to $P_1^{(x)}y^{(y)}$, where P_1 is the x-linear extension of P.*

4. **The generalized Green's lemma.** It is the purpose of this section to prove

THEOREM 13. *If P is defined on R_0, and C, a simple closed curve interior to R_0, is absolutely squarable $(P^{(x)}y^{(y)})$, if K, the interior of C, has inner content $(P^{(x)}y^{(y)})$, and if, moreover, the integral $\int_C Pdy$ exists,* then*

$$\int_C Pdy = \mathrm{cont}_{P^{(x)}y^{(y)}}K = \mathrm{cont}_{P^{(x)}y^{(y)}}(K + C).$$

Let

$$C: \qquad x = \phi(t),\ y = \psi(t) \qquad (0 \leqq t \leqq 1)$$

be parametric equations of C such that as t varies from 0 to 1, C is described in the positive sense. For brevity write

$$P_0(t) = P[\phi(t), \psi(t)].$$

Now let e be a fixed positive number and π_0 a fixed partition of (01) into intervals $\Delta\pi_0$,

$$\pi_0: \qquad t_0(=0),\ t_1,\ \cdots,\ t_{n-1},\ t_n(=1),$$

and suppose π_0 is such that if $\pi F \pi_0$,† that is, if π is a partition obtained by subdividing some or all of the intervals of π_0, then

$$\left| \int_C Pdy - S_\pi^0 P_0 \Delta\psi \right| \leqq \frac{e}{4},$$

where

$$S_\pi^0 P_0 \Delta\psi = \sum_{\Delta\pi} \tfrac{1}{2}\{P_0(\overline{\Delta\pi}) + P_0(\underline{\Delta\pi})\}\psi(\Delta\pi).$$

Next let us form a partition Π_e of $\mathfrak{P}$ by dividing R_0 into horizontal strips $\rho_1, \cdots, \rho_h$ closed and non-overlapping (except for boundary points) and also into equal vertical strips $\sigma_1, \cdots, \sigma_k$ of the same character in such a way that

---

* It is sufficient to assume the integral exists in the weak sense; that is all that is actually used. (Cf. the author's paper cited above.)

† Loc. cit., p. 492.

(1) the width of each horizontal strip is at most equal to the common width of the vertical strips;

(2) each of the points $(\phi(t_i), \psi(t_i))$ which corresponds to a division point t_i of π_0 is on the common boundary of two adjacent horizontal strips;

(3) the inequality

$$\sum_{\Delta\Pi} | P^{(x)}(\Delta\Pi) | \, y^{(y)}(\Delta\Pi)\epsilon(C, \Delta\Pi) \leqq \frac{e}{2}$$

holds for every partition $\Pi F \Pi_e$;

(4) the inequality

$$\left| \operatorname{cont}_{P^{(x)}y^{(y)}} K - \sum_{\Delta\Pi} P^{(x)}(\Delta\Pi) y^{(y)}(\Delta\Pi) [1 - \epsilon(R_0 - K, \Delta\Pi)] \right| \leqq \frac{e}{4}$$

holds for every partition $\Pi F \Pi_e$.

Let us now consider the intersection K_r of K with the interior of ρ_r. Since it is a region (that is, set of inner points), it can be resolved (uniquely) into a finite or denumerably infinite number of connected regions:

$$K_r = Q_{r1} + Q_{r2} + Q_{r3} + \cdots .$$

We say a region Q_{ri} is of the first kind if its boundary contains an arc of C which has points of intersection with both the upper and the lower boundaries of ρ_r; otherwise Q_{ri} is of the second kind. It is easily shown that for given r there are but a finite number of Q_{ri} of the first kind; suppose the notation so chosen that they are $Q_{r1}, \cdots, Q_{ri_r}$. Let Q_r be the sum of the Q_{ri} of the second kind. Then

$$K_r = Q_r + \sum_{i=1}^{i_r} Q_{ri},$$

where all the Q_{ri} are of the first kind.

It is now easily shown that since $Q_{ri}(i=1, \cdots, i_r)$ is connected, its boundary consists of (1) an arc a'_{ri} of C lying entirely within ρ_r except for its end points, of which the first* lies on the lower boundary of ρ_r and the second on the upper boundary of ρ_r; (2) an arc a''_{ri} of the same character except that its first and second end points are respectively on the upper and lower boundaries of ρ_r; (3) a finite or denumerably infinite number of arcs of C each with its end points on the same boundary line of ρ_r; (4) a finite or denumerably infinite number of segments of the upper and lower boundaries of ρ_r.

* The first end point is the one which corresponds to the smaller value of t.

We next form a certain partition π_1 of (01). To this end let I'_{ri}, I''_{ri} be the t-intervals corresponding to arcs a'_{ri}, a''_{ri}, respectively. If a division point t_i of π_0 is not an end point of some I'_{ri} or I''_{ri}, it is an end point of some I_i which is the t-interval corresponding to an arc of C which lies entirely in some strip ρ_r and has its end points on the same horizontal boundary line of that strip. Now take π_1 as the partition whose points of division are the end points of the intervals I'_{ri}, I''_{ri} and existent intervals I_r.

It can now be proved that if $\Delta\pi_1$ is an interval of π_1, then $\psi(\Delta\pi_1)=\psi(\overline{\Delta\pi_1})-\psi(\underline{\Delta\pi_1})=0$ unless $\Delta\pi_1$ is an I'_{ri} or an I''_{ri}. For then $\Delta\pi_1$ is either (1) an I_r, or (2) between an I_r and an I'_{ri} or I''_{ri}, or (3) between two intervals of types I'_{ri}, I''_{ri}. In case (1), $\psi(\Delta\pi_1)=0$ obviously. The same also holds in case (2); for otherwise $\Delta\pi_1$ would correspond to an arc of C with end points on different horizontal boundary lines of ρ_r. But then this arc would contain an arc lying entirely in some ρ_r and with its end points on different horizontal boundary lines of that ρ_r and would therefore correspond to an I'_{ri} or to an I''_{ri}, that is, $\Delta\pi_1$ would contain some I'_{ri} or some I''_{ri}, contrary to hypothesis. The case (3) is similar, and the conclusion is established.

From what has just been proved, it follows that

$$
\begin{aligned}
S^0_{\pi_1}P_0\Delta\psi &= \sum_{ri}\tfrac{1}{2}\{P_0(\bar{I}'_{ri})+P_0(\underline{I}'_{ri})\}\psi(I'_{ri}) \\
&\quad + \sum_{ri}\tfrac{1}{2}\{P_0(\bar{I}''_{ri})+P_0(\underline{I}''_{ri})\}\psi(I''_{ri}). \qquad (1)
\end{aligned}
$$

But

$$
\begin{aligned}
\psi(\underline{I}'_{ri}) &= \psi(\bar{I}''_{ri}) = \underline{y}_r, \text{ say}, \\
\psi(\bar{I}'_{ri}) &= \psi(\underline{I}''_{ri}) = \bar{y}_r, \text{ say},
\end{aligned}
$$

$$
\psi(I'_{ri}) = -\psi(I''_{ri}) = \Delta y_r, \text{ say}.
$$

Moreover let us write

$$
\begin{aligned}
\phi(\bar{I}'_{ri}) &= \bar{x}'_{ri}, \quad \phi(\underline{I}'_{ri}) = \underline{x}'_{ri}, \\
\phi(\bar{I}''_{ri}) &= \bar{x}''_{ri}, \quad \phi(\underline{I}''_{ri}) = \underline{x}''_{ri}.
\end{aligned}
$$

Then (1) may be written

$$
\begin{aligned}
S_{\pi_1}{}^0P\,\Delta\psi &= \sum_r \Delta y_r \sum_{i=1}^{i_r}\frac{1}{2}\{P(\bar{x}'_{ri}\bar{y}_r)-P(\underline{x}''_{ri}\bar{y}_r)\} \\
&\quad + \sum_r \Delta y_r \sum_{i=1}^{i_r}\frac{1}{2}\{P(\underline{x}'_{ri}\underline{y}_r)-P(\bar{x}''_{ri}\underline{y}_r)\}. \qquad (2)
\end{aligned}
$$

Now let R'_{ri}, R''_{ri} be the rectangles

$$R'_{ri}: \qquad \underline{x}''_{ri} \leqq x \leqq \bar{x}'_{ri}, \quad \underline{y}_r \leqq y \leqq \bar{y}_r;$$
$$R''_{ri}: \qquad \bar{x}''_{ri} \leqq x \leqq \underline{x}'_{ri}, \quad \underline{y}_r \leqq y \leqq \bar{y}_r.$$

Also let $\bar{\xi}'_{rsi}$, $\underline{\xi}'_{rsi}$ be respectively the upper and lower bounds of values of x for points (xy) in the (existent) rectangle $\sigma_s \cdot R'_{ri}$. Then

$$[P(\bar{x}'_{ri}\bar{y}_r) - P(\underline{x}''_{ri}\bar{y}_r)]\Delta y_r = \sum_s [P(\bar{\xi}'_{rsi}\bar{y}_r) - P(\underline{\xi}'_{rsi}\bar{y}_r)]\Delta y_r. \tag{3}$$

Therefore

$$\begin{aligned}\Big|\text{cont}_{P^{(x)}{}_y{}^{(y)}}K &- \sum_r \sum_i [P(\bar{x}'_{ri}\bar{y}_r) - P(\underline{x}''_{ri}\bar{y}_r)]\Delta y_r\Big| \\ &\leqq \Big|\text{cont}_{P^{(x)}{}_y{}^{(y)}}K - \sum_r \sum_i \sum_s [P(\bar{\xi}'_{rsi}\bar{y}_r) - P(\underline{\xi}'_{rsi}\bar{y}_r)] \\ &\quad \cdot [1 - \epsilon(R_0 - Q_{ri}, \ \sigma_s \cdot R'_{ri})]\Delta y_r\Big| + \sum_r \sum_i \sum_s \Big| P(\bar{\xi}'_{rsi}\bar{y}_i) \\ &\quad - P(\underline{\xi}'_{rsi}\bar{y}_i)\Big| \cdot \epsilon(R_0 - Q_{ri}, \ \sigma_s \cdot R'_{ri}) \cdot \Delta y_r \leqq \frac{e}{4} + \frac{e}{2} = \frac{3}{4}e.\end{aligned}$$

Similarly

$$\Big|\text{cont}_{P^{(x)}{}_y{}^{(y)}}K - \sum_r \sum_i [P(\underline{x}'_{ri}\underline{y}_r) - P(\bar{x}''_{ri}\underline{y}_r)]\Delta y_r\Big| \leqq \frac{3}{4}e,$$

so that by (4)

$$|\ \text{cont}_{P^{(x)}{}_y{}^{(y)}}K - S_{\pi_1}{}^0 P_0 \Delta\psi\ | \leqq \tfrac{3}{4}e. \tag{4}$$

But

$$\left|\int_C P dy - S_{\pi_1}{}^0 P_0 \Delta\psi\right| \leqq \tfrac{1}{4}e. \tag{5}$$

From (4) and (5) the theorem follows, since e is arbitrary.

5. **On the existence of the line integral of Green's lemma.** Let the curve C and its parametric representation be as above with the additional requirement that the representation be one-to-one for $0 \leqq t < 1$. Let $P(xy)$ be defined on R_0. We seek sufficient conditions that $\int_C P_0(t)d\psi(t)$ exist, where P_0 is as above. The first such condition is given by

THEOREM 14. *If P is continuous on C, the integral $\int_C P_0(t)d\psi(t)$ exists if $\psi(t)$ is of limited variation, in particular if C is rectifiable.*

For then $P_0(t)$ is continuous and the theorem follows from a well known theorem on the Stieltjes integral.

We next obtain a condition less restrictive on C; to this end we first prove two lemmas.

LEMMA 2. *In order that*

$$\underset{\pi}{\mathrm{L}} \sum_{\Delta\pi}(O_{\Delta\pi}P_0)\,|\,\psi(\Delta\pi)\,| = 0$$

it is sufficient that

(i) *P satisfy the Lipschitz condition*

$$|\,P(x_1y_1) - P(x_2y_2)\,| \leqq A\,|\,x_1 - x_2\,| + B\,|\,y_1 - y_2\,|$$

for every pair of points (x_1y_1), (x_2y_2) on C;

(ii) $$\underset{\pi}{\mathrm{L}}\, B \sum_{\Delta\pi}(O_{\Delta\pi}\psi)\,|\,\psi(\Delta\pi)\,| = 0\,;$$

and

(iii)* $$\underset{\pi}{\mathrm{L}}\, A \sum_{\Delta\pi}(O_{\Delta\pi}\phi)\,|\,\psi(\Delta\pi)\,| = 0.$$

To prove this note that for every partition π of (01) there is a system $(t_{1\pi}^{\Delta\pi}, t_{2\pi}^{\Delta\pi}\,|\Delta\pi, \pi)$ where $t_{1\pi}^{\Delta\pi}$, $t_{2\pi}^{\Delta\pi}$ are in $\Delta\pi$, such that

$$\begin{aligned} 0 &\leqq O_{\Delta\pi}P_0 \leqq 2\left[P_0(t_{1\pi}^{\Delta\pi}) - P_0(t_{2\pi}^{\Delta\pi})\right] \\ &\leqq 2A\,|\,\phi(t_{1\pi}^{\Delta\pi}) - \phi(t_{2\pi}^{\Delta\pi})\,| + 2B\,|\,\psi(t_{1\pi}^{\Delta\pi}) - \psi(t_{2\pi}^{\Delta\pi})\,| \\ &\leqq 2\left[A\,O_{\Delta\pi}\phi + B\,O_{\Delta\pi}\psi\right]. \end{aligned}$$

Hence

$$\begin{aligned} 0 &\leqq \sum_{\Delta\pi}(O_{\Delta\pi}P_0)\,|\,\psi(\Delta\pi)\,| \\ &\leqq \left[A\sum_{\Delta\pi}(O_{\Delta\pi}\phi)\,|\,\psi(\Delta\pi)\,| + B\sum_{\Delta\pi}(O_{\Delta\pi}\psi)\,|\,\psi(\Delta\pi)\,|\right]. \end{aligned}$$

Therefore

$$\underset{\pi}{\mathrm{L}} \sum_{\Delta\pi}(O_{\Delta\pi}P_0)\,|\,\psi(\Delta\pi)\,| = 0,$$

as was to be proved.

LEMMA 3. *The condition*

$$\underset{\pi}{\mathrm{L}} \sum_{\Delta\pi}(O_{\Delta\pi}P_0)\,|\,\psi(\Delta\pi)\,| = 0$$

* The condition (iii) is equivalent to the same condition with A omitted if $A \neq 0$; but it is desired to include the case when P is independent of y, in which case A may be taken to be 0; the condition is then satisfied for all ψ. A similar remark applies to (ii).

is sufficient for the existence of $\int_0^1 P_0(t)d\psi(t)$ *provided that* P *is continuous on* C *and that* $P(x'y) \leqq P(x''y)$ *for every pair of points* $(x'y)$, $(x''y)$ *on* C *such that* $x' < x''$.

For then the functions $P_0(t)$, $\psi(t)$ satisfy one of the sufficient conditions of Corollary 1, p. 505 of the author's paper cited above.

We can now state the desired condition as

THEOREM 15. *The integral* $\int_0^1 P_0(t)d\psi(t)$ *exists if*
(i) $P_x(xy)$ *exists and is less in absolute value than a fixed number* N *on* R_0, *and for each* y, $P_x(xy)$ *is a continuous function of* x;
(ii) P, P_x *satisfy the Lipschitz conditions*

$$| P(xy_1) - P(xy_2) | \leqq C | y_1 - y_2 | ,$$

$$| P_x(xy_1) - P_x(xy_2) | \leqq D | y_1 - y_2 |$$

for every pair of points (xy_1), (xy_2) *on* R_0;

(iii) $$\underset{\pi}{\mathrm{L}}(C + D) \sum_{\Delta\pi}(O_{\Delta\pi}\psi) | \psi(\Delta\pi) | = 0 ;$$

(iv) $$\underset{\pi}{\mathrm{L}} N \sum_{\Delta\pi}(O_{\Delta\pi}\phi) | \psi(\Delta\pi) | = 0.$$

To prove this let us form the functions

$$P'(xy) = P(ay) + \frac{1}{2}\int_a^x \{ | P_x(uy) | + P_x(uy)\} du,$$

$$P''(xy) = \frac{1}{2}\int_a^x \{ | P_x(uy) | - P_x(uy)\} du.$$

Clearly

(6) $$P(xy) = P'(xy) - P''(xy)$$

and

(7) $$P'(x'y) \leqq P'(x''y), \quad P''(x'y) \leqq P''(x''y) \qquad (x' < x'').$$

Now consider the function $P'(xy)$. We have

(8) $$| P'(x_1y) - P'(x_2y) | = \frac{1}{2}\left| \int_{x_1}^{x_2} \{ | P_x(uy) | + P_x(uy)\} du \right| \leqq N | x_1 - x_2 | .$$

Also

$$| P'(xy_1) - P'(xy_2) | \leq | P(ay_1) - P(ay_2) | + \frac{1}{2} \left| \int_a^x \{ | P_x(uy_1) | \right.$$

$$\left. - | P_x(uy_2) | + P_x(uy_1) - P_x(uy_2) \} du \right|$$

$$\leq | P(ay_1) - P(ay_2) | + \int_a^x \left| P_x(uy_1) - P_x(uy_2) \right| du \tag{9}$$

$$\leq C | y_1 - y_2 | + \int_a^x D | y_1 - y_2 | du$$

$$\leq C | y_1 - y_2 | + (x - a) D | y_1 - y_2 |$$

$$\leq E | y_1 - y_2 | ,$$

where $E = C + (b-a)D$.

From (8) and (9) we get

$$| P'(x_1y_1) - P'(x_2y_2) | \leq | P'(x_1y_1) - P'(x_1y_2) | + | P'(x_1y_2) - P'(x_2y_2) | \leq N | x_1 - x_2 | + E | y_1 - y_2 | . \tag{10}$$

If we write

$$P_0'(t) = P'[\phi(t),\ \psi(t)],$$

we see, from (10) and the hypothesis, that the conditions of Lemma 2 are satisfied* and hence

$$\underset{\pi}{\mathrm{L}} \sum_{\Delta\pi} (O_{\Delta\pi} P_0') \psi(\Delta\pi) = 0 . \tag{11}$$

But (11) and (7) show (Lemma 3) that $\int_0^1 P_0'(t) d\psi(t)$ exists. Similarly if $P_0''(t) = P''[\phi(t), \psi(t)]$, it can be shown that $\int_0^1 P_0''(t) d\psi(t)$ exists. Hence, by (6), $\int_0^1 P_0(t) d\psi(t)$ exists and equals $\int_0^1 P_0'(t) d\psi(t) - \int_0^1 P_0''(t) d\psi(t)$ and the theorem is proved.

6. **Two special cases.** We can now state two important special cases of Green's lemma. The first is given by

THEOREM 16. *If $P(xy)$ is defined and continuous on a simple closed rectifiable curve C and is defined and possesses a bounded integrable partial derivative $P_x(xy)$ on K, the interior of C, then $\int_C P(xy)dy$ exists and*

$$\int_C P(xy) dy = \iint_K P_x(xy) dx dy . \dagger$$

* The continuity of P in x and y together follows from (i) and (ii) which imply respectively that P is continuous in x for every y and in y uniformly as to x.

† This is the result obtained by Gross, except that he assumes $P(xy)$ to be summable instead of Riemann integrable.

This theorem follows from Theorems 12, 13, 14.

The second is given by

THEOREM 17. *If $P(xy)$ is defined on R_0 and possesses a partial derivative $P_x(xy)$ on the interior of R_0 and if C is a simple closed squarable curve interior to R_0, and K is its interior, then*

$$\int_C P(xy)dy = \int\int_K P_x(xy)dxdy,$$

provided $\int_C P(xy)dy$ exists and $P_x(xy)$ is bounded and integrable on R_0. in particular, provided

(i) *$P_x(xy)$ is, for each y, a continuous function of x;*

(ii) *P and P_x satisfy the Lipschitz conditions*

$$| P(xy_1) - P(xy_2) | \leqq C | y_1 - y_2 | ,$$

$$| P_x(xy_1) - P_x(xy_2) | \leqq D | y_1 - y_2 | ,$$

for every pair of points (xy_1), (xy_2) in R_0;

(iii) $$\underset{\pi}{\mathrm{L}}(C + D) \sum_{\Delta\pi}(O_{\Delta\pi}\psi) | \psi(\Delta\pi) | = 0 ;$$

(iv) $$\underset{\pi}{\mathrm{L}} N \sum_{\Delta\pi}(O_{\Delta\pi}\phi) | \psi(\Delta\pi) | = 0,$$

where N is the upper bound of $|P_x|$ on R_0.

This theorem follows from Theorems 8, 9, 13, 15, since the hypothesis implies the continuity in x and y together of P and P_x (see first footnote, p. 418).

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ON BELL'S ARITHMETIC OF BOOLEAN ALGEBRA*

BY

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1. **Introduction.** Bell† has constructed an arithmetic for an algebra of logic or (following the terminology of Sheffer‡) a Boolean algebra, which presents gratifying analogies to the arithmetics of rational and other fields. In only one detail is the similarity less close than seems appropriate to the difference in the structures of the algebras themselves—namely, in the properties of the concept *congruence*. In this note I shall show that a slightly more general definition retains all the properties of the congruence given by Bell and restores several analogies to rational arithmetic lost by the Bell definition.

All the notation and terminology of the paper of Bell (to which reference should be made for results not here repeated) other than those relating to congruence are followed in the present treatment. In particular, we utilize the two (dual) interpretations of arithmetic operations and relations in $\mathfrak{L}$:

	Name	Symbol	Interpretation I	Interpretation II
(s)	Arithmetic sum:	$\alpha s\beta$:	$\alpha+\beta$,	$\alpha\beta$;
(p)	Arithmetic product:	$\alpha p\beta$:	$\alpha\beta$,	$\alpha+\beta$;
(g)	G. C. D.:	$\alpha g\beta$:	$\alpha+\beta$,	$\alpha\beta$;
(l)	L. C. M.:	$\alpha l\beta$:	$\alpha\beta$,	$\alpha+\beta$;
(ζ)	Arithmetic zero:	ζ:	ω,	ϵ;
(υ)	Arithmetic unity:	υ:	ϵ,	ω;
(d)	α divides β:	$\alpha d\beta$:	$\alpha \mid \beta$,	$\beta \mid \alpha$.

2. **Residuals.** It will be convenient to amplify slightly Bell's treatment of residuals. By the residual of b with respect to a in $\mathfrak{A}$, bra, is meant the quotient of a by the G. C. D. of a and b. Transforming this into a form equivalent for $\mathfrak{A}$ and suitable, by the non-appearance of the concept quotient, for analogy in $\mathfrak{L}$, Bell uses for $\mathfrak{L}$ substantially the following: $\beta r\alpha$ is the G. C. D. of all λ such that α divides the arithmetic product of λ and β. If we use interpretation I, we have that $\beta r\alpha$ is the algebraic (in this case also arithmetic) sum of all λ such that $\alpha \mid \lambda\beta$. But for any such λ, $\lambda\beta\alpha=\lambda\beta$,

* Presented to the Society, April 7, 1928; received by the editors August 20, 1927.

† E. T. Bell, *Arithmetic of logic*, these Transactions, vol. 29 (1927), pp. 597–611.

‡ H. M. Sheffer, *A set of five independent postulates for Boolean algebras with applications to logical constants*, these Transactions, vol. 14 (1913), pp. 481–488.

for which it is necessary and sufficient that $\lambda=\xi(\alpha+\beta')$, where ξ is any element of $\mathfrak{L}$. The sum of all such λ is $\alpha+\beta'$.

If we adopt interpretation II, we find similarly that the residual of β with respect to α is $\alpha\beta'$. We may thus add to the table of interpretations

(r) Residual: $\beta r\alpha$: $\alpha+\beta'$, $\alpha\beta'$.

For both interpretations we may write

$$\beta r\alpha = \alpha s\beta'.$$

3. **Congruence.** In rational number theory the assertion $a\equiv b \bmod m$ means that $a-b$ is divisible by m. If we desire to remain within the set of non-negative rational integers, we may say that $a\equiv b \bmod m$ if there exist c, x, y such that $a=c+mx$, $b=c+my$. We adopt correspondingly as the definition for $\mathfrak{L}$, in place of Bell's (1.1)-(1.7):

$\alpha\equiv\beta \bmod \mu$ if there exist γ, ξ, η such that $\alpha=\gamma s(\mu p\xi)$, $\beta=\gamma s(\mu p\eta)$.

We shall see that this definition satisfies Bell's (1.1)-(1.4) and the Boolean analogies of his (1.5), (1.6) just as do his own interpretations (4.1), (4.2), gives even a better analogy for his (1.7), and preserves several other important analogies with rational arithmetic which otherwise fail.

Under interpretation I we have for $\alpha\equiv\beta \bmod \mu$

$$\alpha = \gamma + \mu\xi, \quad \beta = \gamma + \mu\eta.$$

Multiplying (algebraically) by μ', we find $\alpha\mu'=\gamma\mu'$, $\beta\mu'=\gamma\mu'$; hence $\alpha\mu'=\beta\mu'$. But conversely this condition is sufficient for $\alpha\equiv\beta \bmod \mu$; for if $\alpha\mu'=\beta\mu'$, we may choose $\gamma=\alpha\mu'=\beta\mu'$, $\xi=\alpha$, $\eta=\beta$.

Under interpretation II we find similarly that for $\alpha\equiv\beta \bmod \mu$ it is necessary and sufficient that $\alpha+\mu'=\beta+\mu'$. We therefore replace Bell's interpretations by the following:

(c) $\alpha \equiv \beta \bmod \mu$: $\alpha\mu' = \beta\mu'$, $\alpha + \mu' = \beta + \mu'$.

In both cases, $\alpha\equiv\beta \bmod \mu$ *if and only if* $\alpha p\mu'=\beta p\mu'$.

4. **Satisfaction of Bell's conditions.** The first four of Bell's conditions (1.1)-(1.4), stated directly for $\mathfrak{L}$ in terms of the congruence notation, are as follows:

If $\alpha\equiv\beta \bmod \mu$, then $\beta\equiv\alpha \bmod \mu$.

If $\alpha\equiv\beta \bmod \mu$ and $\beta\equiv\gamma \bmod \mu$, then $\alpha\equiv\gamma \bmod \mu$.

If $\alpha\equiv\beta \bmod \mu$ and $\gamma\equiv\delta \bmod \mu$, then $\alpha s\gamma\equiv\beta s\delta \bmod \mu$.

If $\alpha\equiv\beta \bmod \mu$ and $\gamma\equiv\delta \bmod \mu$, then $\alpha p\gamma\equiv\beta p\delta \bmod \mu$.

That these hold under our criterion $\alpha p\mu'=\beta p\mu'$ is evident.

Bell's statement of (1.5) has as its Boolean analogue:

$\alpha \equiv \zeta \bmod \mu$ if and only if μ divides α.

Testing in interpretation I, we have that $\alpha\mu' = \omega$ if and only if $\mu | \alpha$; that is, $\alpha\mu' = \omega$ if and only if $\alpha\mu = \alpha$, which is true.

The Boolean analogue of Bell's (1.6) is the following:

If $\kappa\alpha \equiv \kappa\beta \bmod \mu$, then $\alpha \equiv \beta \bmod \kappa r \mu$.

Under interpretation I, the hypothesis is $(\kappa\alpha)\mu' = (\kappa\beta)\mu'$, and the conclusion $\alpha(\mu + \kappa')' = \beta(\mu + \kappa')'$; but these statements are identical.

Bell's last condition (1.7) is intended to furnish the analogue of the following in rational arithmetic: $a \equiv a \bmod m$. Such an analogue holds, under Bell's (4.1, 4.2), *only* in the special form (equivalent in the rational case) $0 \equiv 0 \bmod m$. But obviously with the definition of the present paper, we have the *complete* analogue

$$\alpha \equiv \alpha \bmod \mu.$$

In close relationship to this result lies the fact that under Bell's form of congruence no two elements of a Boolean algebra can be congruent unless each is congruent to the arithmetic zero. Indeed, we may compare the generality of the two ideas by observing that while our definition makes $\alpha \equiv \beta \bmod \mu$ if and only if $\alpha p \mu' = \beta p \mu'$, Bell's definition makes $\alpha \equiv \beta \bmod \mu$ if and only if $\alpha p \mu' = \beta p \mu' = \zeta$; the latter thus singles out *one* of the residue classes into which we shall in the next section distribute all the elements of a Boolean algebra.

5. **Residue classes.** In rational number theory (with positive and negative integers) a and b belong to the same residue class with respect to m if $a \equiv b \bmod m$. The residue class of an element a contains all elements x which can be written in the form $x = a + my$. If we restrict ourselves to non-negative integers we may say that the residue class of a consists of all x such that $x = a + my$ and all x such that $a = x + my$. We may then naturally call an element a of a residue class the generator of the class if every member x of the class can be written in the form $x = a + my$. We shall say similarly for $\mathfrak{L}$: the *residue class* of α consists of all ξ such that $\xi = \alpha s(\mu p \eta)$ and all ξ such that $\alpha = \xi s(\mu p \eta)$; α is the *generator* of its residue class if and only if every member ξ of the class can be written in the form $\xi = \alpha s(\mu p \eta)$. The following theorems are then analogues of theorems in the non-negative rational case:

Every residue class with respect to a modulus μ possesses one and only one generator.

$\alpha \equiv \beta \bmod \mu$ *if and only if α and β belong to the same residue class with respect to the modulus μ.*

We shall confine the proofs to interpretation I. To prove the first theorem, let α be any member of a residue class; then $\alpha_0=\alpha\mu'$ is a generator. For if either $\xi=\alpha+\mu\eta$ or $\alpha=\xi+\mu\eta$, then $\xi\mu'=\alpha\mu'$, and $\xi=\alpha_0+\mu\xi$. There can not be two distinct generators α_0, α_1; for if $\alpha_1=\alpha_0+\mu\eta_0$ and $\alpha_0=\alpha_1+\mu\eta_1$, it follows that $\alpha_1|\alpha_0$ and $\alpha_0|\alpha_1$, so that $\alpha_1=\alpha_0$.

The second theorem is obvious, since the generators of the residue classes of α and β, which are respectively $\alpha\mu'$ and $\beta\mu'$, will be equal if and only if $\alpha\equiv\beta \bmod \mu$.

It is clear that the elements of a Boolean algebra which can participate in Bell's definition of congruence are those belonging to the single residue class whose generator is ζ.

6. **Coprimality.** Two elements α, β of a Boolean algebra are called *coprime* (Bell, (22.1)) when $\alpha g\beta=v$. We may express this in terms of congruence (without any precise analogue in rational arithmetic): *α and β are coprime if and only if $\alpha\equiv v \bmod \beta$.* For the definition of coprimality, $\alpha g\beta=v$, is the same as $\alpha s\beta=v$, which is equivalent to $\alpha p\beta'=vp\beta'$ or $\alpha\equiv v$ mod β.

7. **The linear congruence; arithmetic reciprocals.** It is remarkable that while algebraic division (i.e., solution of linear equation) in $\mathfrak{L}$ is nearly always impossible or non-unique, arithmetic division with respect to a modulus is unique under the same hypotheses as in rational arithmetic and possible under the same hypotheses as in rational arithmetic.

If α, μ are coprime, there exists one and (congruentially) only one ξ such that $\alpha p\xi\equiv\beta \bmod \mu$.

For $\alpha\equiv v \bmod \mu$, by the preceding section, and $\xi\equiv\xi \bmod \mu$; thus the given congruence is equivalent to $\xi\equiv\beta \bmod \mu$.

As a special case we have the following:

If α, μ are coprime, then α has with respect to the modulus μ one and (congruentially) only one reciprocal.

The value of the reciprocal is given by $\xi\equiv\alpha\equiv v \bmod \mu$.

For the case of more general α, μ we have the following theorem:

The congruence $\alpha p\xi\equiv\beta \bmod \mu$ *has no solution unless* $(\alpha g\mu)\beta d$; *if this condition holds, and*

$$\alpha g\mu = \delta, \qquad \delta r\mu = \mu_1, \qquad \alpha = \delta p\alpha_1, \qquad \beta = \delta p\beta_1,$$

then every solution is given by

$$\xi \equiv \xi_1 s(\eta p\mu_1) \bmod \mu,$$

where ξ_1 *is a properly selected element of the algebra satisfying the congruence* $\alpha_1 p\xi_1 \equiv \beta_1 \bmod \mu_1$, *and* η *is arbitrary.*

We give the proof under interpretation I. Let

$$\alpha\xi \equiv \beta \bmod \mu, \tag{1}$$

$$\delta = \alpha + \mu. \tag{2}$$

Then

$$\alpha\xi\mu' = \beta\mu', \qquad \beta = \alpha\xi\mu' + \beta\mu\ ;$$

since $\delta \mid \alpha$ and $\delta \mid \mu$, it follows that $\delta \mid \beta$.

Now let

$$\mu_1 = \delta' + \mu, \qquad \alpha = \delta\alpha_1, \qquad \beta = \delta\beta_1. \tag{3}$$

The congruence $\alpha_1\xi_1 \equiv \beta_1 \bmod \mu_1$ has a solution; for

$$\begin{aligned} \alpha_1 + \mu_1 &= \alpha_1(\delta + \delta') + (\delta' + \mu) = (\alpha_1\delta + \mu) + (\alpha_1\delta' + \delta') \\ &= (\alpha + \mu) + \delta' = \delta + \delta' = \epsilon, \end{aligned}$$

so that α_1, μ_1 are coprime and $\alpha_1 \equiv \epsilon \bmod \mu_1$. A solution is $\xi_1 = \beta$; for $\alpha_1\xi_1 = \alpha_1\beta = \alpha_1\delta\beta_1 \equiv \epsilon\delta\beta_1 \equiv \delta\beta_1 \equiv \beta \bmod \mu_1$. We shall show that

$$\xi \equiv \beta + \eta\mu_1 \bmod \mu \tag{4}$$

is for every η a solution of (1). By (2), (3), $\alpha \equiv \delta \bmod \mu$, and $\mu_1 \equiv \delta' \bmod \mu$; hence

$$\alpha\xi \equiv \delta\xi \equiv \delta\beta + \eta\delta\mu_1 \bmod \mu,$$

$$\alpha\xi \equiv \beta + \eta\delta\delta' \bmod \mu,$$

$$\alpha\xi \equiv \beta \bmod \mu.$$

Conversely every solution of (1) is of the form (4) for some η. For from (1) and (3) we deduce that $\delta\alpha_1\xi \equiv \delta\beta_1 \bmod \mu$, and by Bell, (1.5),

$$\alpha_1\xi \equiv \beta_1 \bmod \mu_1.$$

Hence $\xi \equiv \beta_1 \bmod \mu_1$, ξ is with respect to μ_1 in the residue class generated by $\beta_1\mu_1'$, and $\xi = \beta_1\mu_1' + \eta\mu_1$. But $\beta_1\mu_1' = \beta_1\delta\mu' = \beta\mu'$, $\beta_1\mu_1'$ is with respect to μ in the residue class generated by $\beta\mu'$, and $\beta_1\mu_1' \equiv \beta \bmod \mu$. Thus (4) must hold.

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A THEOREM ON ORTHOGONAL FUNCTIONS WITH AN APPLICATION TO INTEGRAL INEQUALITIES*

BY
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It is well known that for a given finite set of functions

$$\{f_i\} \qquad f_1(x),\ f_2(x),\ \cdots,\ f_m(x),$$

continuous on an interval

$$(\mathfrak{X}) \qquad a \leqq x \leqq b,$$

a continuous function $f(x)$ can be determined which is orthogonal to all functions of the set, that is, the conditions

$$\int_a^b f(x)f_j(x)dx = 0 \qquad (j = 1, 2, \cdots, m)$$

can be satisfied. The principal object of the present paper is to determine under what conditions such a function $f(x)$ can be everywhere *positive*. This object is attained in the following

THEOREM I. *A necessary and sufficient condition that a set of functions, continuous and linearly independent on a closed interval, admit a positive continuous function orthogonal to all of them is that every linear combination of the functions change sign on the interval.*†

In the last section of the paper an application of this theorem is given in a study of the integral inequality

$$(1) \qquad \phi(x) + \int_a^b \kappa(x,s)\phi(s)ds > 0.$$

The principal result in this connection is the following

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† This theorem is an analogue of an algebraic theorem given in a recent paper, *Note on certain associated systems of linear equalities and inequalities*, Annals of Mathematics, (2), vol. 28 (1926–27), pp. 41–42.

THEOREM II. *A necessary and sufficient condition that the integral inequality* (1) *admit a solution* $\phi(x)$ *is that every non-trivial solution* $\psi(x)$ *of the associated integral equation*

$$\psi(x) + \int_a^b \psi(s)\kappa(s,x)ds = 0$$

*shall change sign.**

The first part of the paper is devoted to preliminary notions which are used in the proof of Theorem I.

1. **An outline of the proof of Theorem I.** The necessity of the condition in Theorem I is almost obvious. For if $f(x)$ is orthogonal to each of the functions $f_i(x)$, it is orthogonal to every linear combination of them, that is

$$\int_a^b f(x) \sum_{i=1}^{m} a_i f_i(x) dx = 0 ;$$

and this relation for a positive $f(x)$ demands that the second factor of the integrand change sign unless it is identically zero. Hence since the functions are linearly dependent, the condition of the theorem is necessary.

To prove the sufficiency of the condition, we proceed as follows. By a certain well defined operation, the given set of m functions $\{f_i\}$ is replaced by another set of m functions called a *reduced* set.† The efficacy of the reduction is due to the following two properties:

(a) One function of the reduced set is identically zero.

(b) If the reduced set admits a positive function orthogonal to all of its members, the same is true of the original set.

The reduction process may be repeated, the property (b) persisting, and the property (a) introducing a new zero function at each repetition; so that after $m-1$ reductions the resulting set contains only one function which is different from zero. It is not difficult to show that this function admits a positive orthogonal function, whence from (b) it follows that the given set admits a positive orthogonal function.

A complication arises from the fact that the range of the argument of a "reduced" set of functions is not the same as the range of the argument of the original set. The functions of the reduced set are as a matter of fact functions of a variable whose range is a composite range.

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* For an analogous algebraic theorem, see papers by Carver, Annals of Mathematics, (2), vol. 23, p. 212; and the author, ibid., vol. 27, p. 57.

† A reduction process for a set of functions on a general range has been described by the author in an earlier paper, *On sets of functions of a general variable*, these Transactions, vol. 29 (1927), pp. 463–470.

2. **Reduction and composition of a range relative to a function on it.** Let us consider any continuous function $\rho(x)$ on the range

$$(\mathfrak{X}) \qquad a \leqq x \leqq b,$$

which changes sign on the range. Relative to the function $\rho(x)$, we may determine two subclasses of the range $\mathfrak{X}$, defined as follows:

$$\mathfrak{X}_P^{(\rho)} \equiv [\text{all } x \text{ such that } \rho(x) \geqq 0],$$

$$\mathfrak{X}_N^{(\rho)} \equiv [\text{all } x \text{ such that } \rho(x) < 0].$$

For convenience we denote the elements of these subclasses by $p^{(\rho)}$ and $n^{(\rho)}$ respectively; thus

$$(2) \qquad \mathfrak{X}_P^{(\rho)} \equiv [p^{(\rho)}], \quad \mathfrak{X}_N^{(\rho)} \equiv [n^{(\rho)}].$$

From the properties of continuous functions we may draw the following conclusions with reference to these subclasses. The subclass $\mathfrak{X}_N^{(\rho)}$ consists (geometrically speaking) of linear intervals open at both ends except when terminated by an end point a or b of the fundamental interval $\mathfrak{X}$. The subclass $\mathfrak{X}_P^{(\rho)}$ is a closed point set. If the reducing function $\rho(x)$ is of simple type (for example if it changes sign at each point at which it vanishes), $\mathfrak{X}_P^{(\rho)}$ consists of closed linear intervals; and it will in fact always contain at least one such interval.*

From the two subclasses (2) we now form a composite range

$$\mathfrak{X}^{(\rho)} \equiv \mathfrak{X}_P^{(\rho)}\mathfrak{X}_N^{(\rho)},$$

the elements of which are bipartite, of the form $p^{(\rho)}n^{(\rho)}$. The new range

$$\mathfrak{X}^{(\rho)} \equiv [x^{(\rho)}] \equiv [p^{(\rho)}n^{(\rho)}]$$

is a two-dimensional point set contained in the square $\mathfrak{X}\mathfrak{X}$. If $\rho(x)$ is of a simple type, the points constitute a set of rectangles forming a sort of irregular checker-board arrangement. And for any $\rho(x)$, the range will include at least one such rectangle.

A function on the range $\mathfrak{X}^{(\rho)}$ will be said to be *continuous* if it is a continuous function of the two variables $p^{(\rho)}$, $n^{(\rho)}$. A function will be said to

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* But since $\mathfrak{X}_P^{(\rho)}$ includes the zeros of $\rho(x)$, it may contain isolated points, and even perfect sets of points which comprise no interval, as in the following example. Let $\mathfrak{X}$ be the closed interval (0, 2). On the left half of this interval form the Cantor perfect set by the removal of open middle thirds (see Pierpont's *Theory of Functions of Real Variables*, vol. I, § 272). On this perfect set let $\rho(x)=0$, and on its complement with respect to (0, 1) let $\rho(x)$ be negative. For $x>1$ let $\rho(x)$ be positive. This interesting example was suggested to me by Professor Kellogg, to whom I am indebted for a number of valuable criticisms and suggestions.

change sign internally on the range $\mathfrak{X}^{(\rho)}$ if it is *positive* at some *inner* points and *negative* at other *inner* points of the range. Analogous definitions of continuity and internal change of sign are to be understood relative to all the composite ranges occurring in what follows.

Suppose now that on the new composite range $\mathfrak{X}^{(\rho)}$ there is defined in any way a real, single-valued, continuous function σ, which changes sign internally on the range. Then this function determines two subclasses of the range $\mathfrak{X}^{(\rho)}$:

$$\mathfrak{X}_P{}^{(\rho\sigma)} \equiv [\text{all } x^{(\rho)} \text{ for which } \sigma \geqq 0],$$

$$\mathfrak{X}_N{}^{(\rho\sigma)} \equiv [\text{all } x^{(\rho)} \text{ for which } \sigma < 0] ;$$

and from these subclasses we may form a composite class

$$\mathfrak{X}^{(\rho\sigma)} \equiv \mathfrak{X}_P{}^{(\rho\sigma)}\mathfrak{X}_N{}^{(\rho\sigma)}.$$

The elements of this class are quadripartite. Geometrically they form a point set in the four-dimensional hypercube $\mathfrak{X}\mathfrak{X}\mathfrak{X}\mathfrak{X}$, of which some subset at least are inner points.

The process of reduction and composition can be repeated indefinitely, provided at each stage a reducing function is available. To generalize the procedure and notation, suppose reduction with respect to the successive reducing functions $\rho_1, \rho_2, \cdots, \rho_{k-1}$ has yielded the composite range

$$\mathfrak{X}^{(\rho_1\rho_2\cdots\rho_{k-1})},$$

consisting of 2^{k-1}-partite elements. Suppose further that ρ_k is a single-valued continuous function changing sign internally on this range. By reduction with respect to ρ_k and composition we obtain the new range

$$\mathfrak{X}^{(\rho_1\rho_2\cdots\rho_k)} \equiv \mathfrak{X}_P{}^{(\rho_1\rho_2\cdots\rho_k)}\mathfrak{X}_N{}^{(\rho_1\rho_2\cdots\rho_k)} \equiv [x^{(\rho_1\rho_2\cdots\rho_k)}],$$

the elements of which are 2^k-partite. Geometrically the new range is a point set in the 2^k-dimensional cube $\mathfrak{X}\mathfrak{X}\cdots\mathfrak{X}$, of which some points at least are inner points.

3. **Reduced outer multiplication.** Consider again the original range $\mathfrak{X}$ and a continuous reducing function $\rho(x)$ changing sign on $\mathfrak{X}$. This function determines with any second continuous function $f(x)$ a function on the composite range $\mathfrak{X}^{(\rho)}$, which we shall call their reduced outer product and denote by $((\rho f))$. Its functional values are defined by the formula

$$((\rho f)) \qquad \rho(p)f(n) - f(p)\rho(n), \quad (p, n) \text{ on } \mathfrak{X}_P{}^{(\rho)}\mathfrak{X}_N{}^{(\rho)}.$$

This multiplication is clearly not commutative. Its most obvious property is that $((\rho\rho))=0$. Another property easily verified is that $((\rho f))$ is continuous on $\mathfrak{X}_{P}^{(\rho)}\,\mathfrak{X}_{N}^{(\rho)}$.

Reduced outer multiplication is defined in a similar way upon any of the composite ranges described in the preceding section. Consider for example the reduced composite range

$$\mathfrak{X}^{(\rho_1\rho_2\cdots\rho_{k-1})},$$

and suppose that ρ_k is a continuous function changing sign on this range. Then ρ_k determines with any second continuous function f_k on the range a reduced outer product $((\rho_k f_k))$, given by the formula

$$((\rho_k f_k)) \qquad \rho_k(p)f_k(n) - f_k(p)\rho_k(n), \qquad (p,n) \text{ on } \mathfrak{X}_{P}^{(\rho_1\rho_2\cdots\rho_k)}\mathfrak{X}_{N}^{(\rho_1\rho_2\cdots\rho_k)}.$$

This product is continuous at all points of its region of definition.

4. **Reduction of a set of functions.** Consider the set of functions

$$\{f_i\} \qquad f_1(x),\, f_2(x),\, \cdots,\, f_m(x),$$

continuous on the range $\mathfrak{X}$, and suppose that $f_1(x)$ changes sign.

Relative to $f_1(x)$ we form a new set of m functions

$$\{f_i^{(1)}\} \qquad ((f_1f_1)),\, ((f_1f_2)),\, \cdots,\, ((f_1f_m)),$$

each of which is the reduced outer product of the corresponding function of the given set by f_1. All functions of this reduced set are on the composite range $\mathfrak{X}^{(f_1)}$, and are continuous on that range. It is to be noted particularly that the first function $((f_1f_1))$ is identically zero. (See property (a) of §1.)

The reduction process may now be repeated, the second function $f_2^{(1)}$ $(\equiv((f_1f_2)))$ being used as a reducing function (assuming that it changes sign), and a second reduced set of functions obtained, having the property that its first two functions are identically zero. Its range $\mathfrak{X}^{(f_1f_2^{(1)})}$ will be denoted for brevity by $\mathfrak{X}^{(12)}$.

To make the reduction procedure and notation general, let $\{f_i^{(12\cdots k-1)}\}$ denote the set of functions obtained by $k-1$ successive reductions of the kind indicated. Its first $k-1$ functions are identically zero, and its kth function is $f_k^{(12\cdots k-1)}$. If this function changes sign we may use it as a reducing function and form the new set $\{f_i^{(12\cdots k)}\}$, each function of which is the reduced outer product of the corresponding function of $\{f_i^{(12\cdots k-1)}\}$ by $f_k^{(12\cdots k-1)}$. The functions of this new set are defined and continuous on the range $\mathfrak{X}^{(12\cdots k)}$, and the first k of them are identically zero.

The reduction procedure thus defined will after $m-1$ operations yield a set of functions all of which except the last are identically zero. The assumption which we have made that at each stage the reducing function changes sign will now be justified.

LEMMA I. *If every linear combination*

$$a_1f_1(x) + a_2f_2(x) + \cdots + a_mf_m(x)$$

of the set of functions $\{f_i(x)\}$ *changes sign on* $\mathfrak{X}$, *then every linear combination*

$$a_2f_2^{(1)} + a_3f_3^{(1)} + \cdots + a_mf_m^{(1)}$$

of the last $m-1$ *functions of the reduced set* $\{f_i^{(1)}\}$ *changes sign internally on its range* $\mathfrak{X}^{(1)}$.

To prove the proposition indirectly, suppose that there is a set of constant multipliers $a_2, a_3, \cdots, a_m$, such that

$$\sum_{j=2}^{m} a_jf_j^{(1)} \geqq 0 \qquad \text{on } \mathfrak{X}^{(1)}. \tag{3}$$

We first recall that the range $\mathfrak{X}^{(1)}$ is a composite range, $\mathfrak{X}^{(1)} \equiv \mathfrak{X}_P^{(1)}\mathfrak{X}_N^{(1)}$, the first component class $\mathfrak{X}_P^{(1)}$ consisting of those elements of $\mathfrak{X}$ for which f_1 is positive *or zero*. For our present purpose, it is convenient to divide the class $\mathfrak{X}_P^{(1)}$ into two subclasses:

$$\mathfrak{X}_P^{(1)} \equiv \mathfrak{X}_{P'}^{(1)} + \mathfrak{X}_Z^{(1)},$$

where

$$\mathfrak{X}_{P'}^{(1)} \equiv [\text{all } x \text{ for which } f_1 > 0],$$

$$\mathfrak{X}_Z^{(1)} \equiv [\text{all } x \text{ for which } f_1 = 0].$$

Next, recalling the definition of the reduced function $f_j^{(1)}$, we may replace our supposition (3) by the two statements

$$\sum_{j=2}^{m} a_j[f_1(p')f_j(n) - f_j(p')f_1(n)] \geqq 0 \qquad (p', n) \text{ on } \mathfrak{X}_{P'}^{(1)}\mathfrak{X}_N^{(1)}, \tag{4}$$

$$-\sum_{j=2}^{m} a_jf_j(z)f_1(n) \geqq 0 \qquad (z, n) \text{ on } \mathfrak{X}_Z^{(1)}\mathfrak{X}_N^{(1)}. \tag{5}$$

Now since $-f_1(p')f_1(n)$ is certainly positive, we may obtain from (4) the equivalent statement

$$(6) \qquad \sum_{j=2}^{m} a_j \frac{f_j(p')}{f_1(p')} \geqq \sum_{j=2}^{m} a_j \frac{f_j(n)}{f_1(n)} \qquad p' \text{ on } \mathfrak{X}_{P'}^{(1)},\ n \text{ on } \mathfrak{X}_N^{(1)}.$$

The various values of the expression on the left side of (6) must have a greatest lower bound, and those of the expression on the right must have a least upper bound, which bounds may or may not coincide.

In any case we may choose a constant a_1 such that

$$\sum_{j=2}^{m} a_j \frac{f_j(p')}{f_1(p')} \geqq -a_1 \geqq \sum_{j=2}^{m} a_j \frac{f_j(n)}{f_1(n)} \qquad p' \text{ on } \mathfrak{X}_{P'}^{(1)},\ n \text{ on } \mathfrak{X}_N^{(1)}.$$

And from this double relation we obtain

$$(7) \qquad \sum_{j=1}^{n} a_j f_j(p') \geqq 0, \qquad \sum_{j=1}^{n} a_j f_j(n) \geqq 0 \qquad p' \text{ on } \mathfrak{X}_{P'}^{(1)},\ n \text{ on } \mathfrak{X}_N^{(1)}.$$

Furthermore, division of (5) by $-f_1(n)$, which is certainly positive, yields a statement which may be written

$$(8) \qquad \sum_{j=1}^{m} a_j f_j(z) \geqq 0 \qquad z \text{ on } \mathfrak{X}_Z^{(1)}.$$

But the statements (7) and (8) can be combined to give

$$\sum_{j=1}^{m} a_j f_j(x) \geqq 0 \qquad x \text{ on } \mathfrak{X}.$$

This contradicts the hypothesis of the lemma, and the contradiction proves that every linear combination $\sum_{j=2}^{m} a_j f_j^{(1)}$ must be negative somewhere on $\mathfrak{X}^{(1)}$. To see that it must be negative at an inner point, we note first that all points of $\mathfrak{X}'_{P}{}^{(1)} \mathfrak{X}_N^{(1)}$ are inner points. If it is negative at none of these points, then it must be negative on $\mathfrak{X}_Z^{(1)} \mathfrak{X}_N^{(1)}$, that is, the left side of (5) must be somewhere negative, and hence the left side of (8) must be negative at some point of $\mathfrak{X}_Z^{(1)}$. If the left side of (8) is negative at an inner point of $\mathfrak{X}_Z^{(1)}$, then the left side of (5) is negative at an inner point of $\mathfrak{X}_Z^{(1)} \mathfrak{X}_N^{(1)}$ which is *a fortiori* an inner point of $\mathfrak{X}^{(1)}$, and our desired conclusion is reached. If on the other hand the left side of (8) is negative only at frontier points of $\mathfrak{X}_Z^{(1)}$, it follows from continuity (since each such frontier point is a limit point of $\mathfrak{X}'_{P}{}^{(1)}$ or $\mathfrak{X}_N^{(1)}$) that one of the inequalities in (7) is contradicted. But this involves a contradiction of (4), which contradiction means explicitly that $\sum_{j=2}^{m} a_j f_j^{(1)}$ is negative on $\mathfrak{X}'_{P}{}^{(1)} \mathfrak{X}_N^{(1)}$, hence at an inner point of $\mathfrak{X}^{(1)}$. In an entirely analogous way it may be shown that every such linear combination must be *positive* at an inner point of $\mathfrak{X}^{(1)}$. The proof of the lemma is then complete.

An argument similar to the one we have just made suffices to prove the following

LEMMA II. *If every linear combination*

$$a_k f_k^{(12\cdots k-1)} + a_{k+1} f_{k+1}^{(12\cdots k-1)} + \cdots + a_m f_m^{(12\cdots k-1)}$$

of the last $m-k+1$ *functions of the reduced set* $\{f_i^{(12\cdots k-1)}\}$ *changes sign on its range* $\mathfrak{X}^{(12\cdots k-1)}$, *then every linear combination*

$$a_{k+1} f_{k+1}^{(12\cdots k)} + a_{k+2} f_{k+2}^{(12\cdots k)} + \cdots + a_m f_m^{(12\cdots k)}$$

of the last $m-k$ *functions of the reduced set* $\{f_i^{(12\cdots k)}\}$ *changes sign internally on its range* $\mathfrak{X}^{(12\cdots k)}$.

Successive application of these lemmas now justifies the assumption which we made in the early part of this section:

If every linear combination of the given set of functions changes sign on $\mathfrak{X}$, *then every function appearing as a reducing function in the progressive reduction process described in this section changes sign internally on its range.*

5. **Proof of property** (b) **of** §1. Two functions f and g defined upon one of the reduced composite ranges $\mathfrak{X}^{(12\cdots k)}$, will be said to be orthogonal one to the other if*

$$\int_{\mathfrak{X}^{(12\cdots k)}} fg = 0.$$

Suppose now that $k-1$ reductions of the given set of functions

$$f_1, f_2, \cdots, f_m$$

have yielded the set

$$0, \cdots, 0, \ f_k^{(12\cdots k-1)}, \ f_{k+1}^{(12\cdots k-1)}, \cdots, f_m^{(12\cdots k-1)},$$

on the range $\mathfrak{X}^{(12\cdots k-1)}$, and the reduction of this latter set with respect to $f_k^{(12\cdots k-1)}$ has yielded the set

$$0, \cdots, 0, \ f_{k+1}^{(12\cdots k)}, \ f_{k+2}^{(12\cdots k)}, \cdots, f_m^{(12\cdots k)},$$

on the range $\mathfrak{X}^{(12\cdots k)}$.

* In this section and in those which follow, the integrals may be taken in the sense of Lebesgue whenever there is doubt as to their existence in the sense of Riemann. Of their existence in the former sense there will be no doubt.

Suppose further that relative to this last set $\{f_j^{(12\cdots k)}\}$ *there is a function* $\Pi^{(12\cdots k)}$, *everywhere positive and continuous on the range* $\mathfrak{X}^{(12\cdots k)}$, *and orthogonal to all functions of the set.*

Then there is a function $\Pi^{(12\cdots k-1)}$, *everywhere positive on the preceding range* $\mathfrak{X}^{(12\cdots k-1)}$, *which is orthogonal to all functions of the preceding set* $\{f_j^{(12\cdots k-1)}\}$.

To prove this proposition we start with the hypothesis that the positive function $\Pi^{(12\cdots k)}$ satisfies the conditions

$$\int_{\mathfrak{X}^{(12\cdots k)}} \Pi^{(12\cdots k)} f_j^{(12\cdots k)} = 0 \qquad (j = 1,2,\cdots,m). \tag{9}$$

Since the defining formula for $f_j^{(12\cdots k)}$ is

$$f_k^{(12\cdots k-1)}(p) f_j^{(12\cdots k-1)}(n) - f_j^{(12\cdots k-1)}(p) f_k^{(12\cdots k-1)}(n)$$
$$(p,n) \text{ on } \mathfrak{X}_P^{(12\cdots k)}\mathfrak{X}_N^{(12\cdots k)},$$

we obtain, by substitution in (9) and decomposition of multiple integrals, the equalities

$$\int_{\mathfrak{X}_N^{(12\cdots k)}} \left[\int_{\mathfrak{X}_P^{(12\cdots k)}} \Pi^{(12\cdots k)}(p,n) f_k^{(12\cdots k-1)}(p)dp\right] f_j^{(12\cdots k-1)}(n)dn$$
$$- \int_{\mathfrak{X}_P^{(12\cdots k)}} \left[\int_{\mathfrak{X}_N^{(12\cdots k)}} \Pi^{(12\cdots k)}(p,n) f_k^{(12\cdots k-1)}(n)dn\right]$$
$$\cdot f_j^{(12\cdots k-1)}(p)dp = 0 \qquad (j = 1,2,\cdots,m).$$

These may be written in form

$$\int_{\mathfrak{X}^{(12\cdots k-1)}} \Pi^{(12\cdots k-1)} f_j^{(12\cdots k-1)} = 0 \qquad (j = 1,2,\cdots,m),$$

the function $\Pi^{(12\cdots k-1)}$ being defined on the range $\mathfrak{X}^{(12\cdots k-1)}$ as follows:

$$\Pi^{(12\cdots k-1)} \equiv \begin{cases} \displaystyle\int_{\mathfrak{X}_P^{(12\cdots k)}} \Pi^{(12\cdots k)}(p,n) f_k^{(12\cdots k-1)}(p)dp \text{ on } \mathfrak{X}_N^{(12\cdots k)}, \\ \displaystyle -\int_{\mathfrak{X}_N^{(12\cdots k)}} \Pi^{(12\cdots k)}(p,n) f_k^{(12\cdots k-1)}(n)dn \text{ on } \mathfrak{X}_P^{(12\cdots k)}. \end{cases}$$

It is obvious from this definition that the function $\Pi^{(12\cdots k-1)}$ is positive on the range $\mathfrak{X}^{(12\cdots k-1)}$. It is likewise continuous at all points except possibly at those points of $\mathfrak{X}_P^{(12\cdots k)}$ which are limit points of $\mathfrak{X}_N^{(12\cdots k)}$.

In the next section we make an alteration in the procedure which will insure continuity at these points also.

6. **An alteration to secure continuity.** In the preceding section the function $\Pi^{(12\cdots k)}$ is assumed to possess certain properties, and upon these assumptions the existence and certain properties of the function $\Pi^{(12\cdots k-1)}$ are established. The function $\Pi^{(12\cdots k)}$ of the hypothesis is still unnecessarily general for our purpose. By making certain additional assumptions with regard to it we may expect to obtain additional properties for the function $\Pi^{(12\cdots k-1)}$. The additional property desired is continuity. To this end, we now assume that $\Pi^{(12\cdots k)}$ is equal to unity at all points of the range $\mathfrak{X}^{(12\cdots k)}$ except on some closed aggregates of inner points.

Now if p_0 is a point of $\mathfrak{X}_P^{(12\cdots k)}$ which is a limit point of $\mathfrak{X}_N^{(12\cdots k)}$, then (p_0, n) is, for every n, a frontier point of $\mathfrak{X}^{(12\cdots k)}$, and from the assumption just made it follows that $\Pi^{(12\cdots k)}(p_0, n)=1$. Hence from the definition of $\Pi^{(12\cdots k-1)}$,

$$\Pi^{(12\cdots k-1)}(p_0) = -\int_{\mathfrak{X}_N^{(12\cdots k)}} f_k^{(12\cdots k-1)}(n)dn.$$

Furthermore, from the assumption it follows that for all points n of $\mathfrak{X}_N^{(12\cdots k)}$ sufficiently near to p_0, $\Pi^{(12\cdots k)}(p, n)=1$, and hence for such values of n

$$\Pi^{(12\cdots k-1)}(n) = \int_{\mathfrak{X}_P^{(12\cdots k)}} f_k^{(12\cdots k-1)}(p)dp.$$

Hence we secure continuity of $\Pi^{(12\cdots k-1)}$ at all such points p_0 on the range $\mathfrak{X}^{(12\cdots k-1)}$ if

$$(10) \qquad \int_{\mathfrak{X}_P^{(12\cdots k)}} f_k^{(12\cdots k-1)}(p)dp = -\int_{\mathfrak{X}_N^{(12\cdots k)}} f_k^{(12\cdots k-1)}(n)dn = 1.$$

We secure this property (10) by prefixing a slight preparatory operation to the reduction process previously described.

In the set

$$\{f_i^{(12\cdots k-1)}\} \qquad 0, \cdots, 0, f_k^{(12\cdots k-1)}, \cdots, f_m^{(12\cdots k-1)},$$

the function $f_k^{(12\cdots k-1)}$, specified as the next reducing function, may or may not satisfy the equality

$$\int_{\mathfrak{X}^{(12\cdots k-1)}} f_k^{(12\cdots k-1)} = 0.$$

If this condition is satisfied, then

$$\int_{\mathfrak{X}_P^{(12\cdots k)}} f_k^{(12\cdots k-1)} = -\int_{\mathfrak{X}_N^{(12\cdots k-1)}} f_k^{(12\cdots k-1)} = \text{a positive constant},$$

and division of $f_k^{(12\cdots k-1)}$ by this constant gives a new function by which it can be replaced, and the new function will have the property indicated by (10).

If the condition (10) is not satisfied by a constant times $f_k^{(12\cdots k-1)}$ but the analogous condition is satisfied by a constant times some succeeding function in the sequence $\{f_j^{(12\cdots k-1)}\}$, the two functions may be interchanged and the desired property secured.

Suppose then that no one of the functions $f_j^{(12\cdots k-1)}$ $(j \geqq k)$ satisfies the condition indicated by (10). Then we replace $f_k^{(12\cdots k-1)}$ by a linear combination

$$\bar{f}_k^{(12\cdots k-1)} = \alpha f_k^{(12\cdots k-1)} + \beta f_{k+1}^{(12\cdots k-1)}$$

where the constants α and β are so chosen that the condition analogous to (10) is satisfied by the function $\bar{f}_k^{(12\cdots k-1)}$. The replacement of $f_k^{(12\cdots k-1)}$ by this function will not affect any of the vital properties of our reduction process, and its use secures the desired continuity of the function $\Pi^{(12\cdots k-1)}$.

Furthermore, a consideration of the definition of $\Pi^{(12\cdots k-1)}$ in the preceding section, together with the additional hypothesis on $\Pi^{(12\cdots k)}$ in the present section and the condition (10), shows that the function $\Pi^{(12\cdots k-1)}$ is equal to unity at all points of its range $\mathfrak{X}^{(12\cdots k-1)}$ except some closed sub-sets composed entirely of inner points of the range. Hence we have the following amplification of the property (b) as stated in §1:

If all the functions of any reduced set admit a common orthogonal function which is positive, continuous, and equal to unity except on some closed sub-sets of inner points, then the set of functions from which it is obtained by reduction has the same property.

7. **Conclusion of the proof of Theorem I.** Starting with the given set of functions

$$\{f_i\} \qquad f_1(x),\, f_2(x),\, \cdots,\, f_m(x),$$

we obtain, after $m-1$ reductions of the type described in the preceding sections, a set

$$0, \cdots, 0, f_m^{(12\cdots m-1)},$$

in which all functions except the last one, $f_m^{(12\cdots m-1)}$, are identically zero. If this last set admits a positive orthogonal function, continuous, and equal to unity except on closed sub-sets of inner points, then by §6 the given set admits a positive and continuous orthogonal function, and the proof of our theorem is complete.

Our problem is then reduced to showing that the set $\{f_i^{(12\cdots m-1)}\}$ admits an orthogonal function of the kind described, or more simply still, that the single function $f_m^{(12\cdots m-1)}$ admits such an orthogonal function. This is not difficult.

From the hypothesis that every linear combination of the given functions changes sign, it follows by §4 that $f_m^{(12\cdots m-1)}$ changes sign internally. Therefore from the continuity preserved in the reduction process it follows that there is a set of inner points completely bounded by inner points on which the function is positive, and a similar set of points on which it is negative. We denote, for the moment, the former set by $\mathfrak{P}$ and the latter set by $\mathfrak{N}$, and consider two functions Π_p and Π_n on the range $\mathfrak{X}^{(12\cdots m-1)}$, restricted by the following conditions. Both functions are continuous. The former Π_p is positive on the region $\mathfrak{P}$ (excluding the boundary) and is elsewhere zero. The latter Π_n is positive on the region $\mathfrak{N}$ (excluding the boundary) and is elsewhere zero.

Next we consider the function

$$\Pi^{(12\cdots m-1)} \equiv 1 + \alpha\Pi_p + \beta\Pi_n, \tag{11}$$

α and β being constants to be determined. This function is evidently continuous on $\mathfrak{X}^{(12\cdots m-1)}$, and is positive if α and β are positive. Furthermore it is equal to unity except at points of $\mathfrak{P}$ and $\mathfrak{N}$.

We now choose α and β so that

$$\int_{\mathfrak{X}^{(12\cdots m-1)}} \Pi^{(12\cdots m-1)} f_m^{(12\cdots m-1)} = 0,$$

which is equivalent to

$$\int_{\mathfrak{X}^{(12\cdots m-1)}} f_m^{(12\cdots m-1)} + \alpha \int_{\mathfrak{X}^{(12\cdots m-1)}} \Pi_p f_m^{(12\cdots m-1)} + \beta \int_{\mathfrak{X}^{(12\cdots m-1)}} \Pi_n f_m^{(12\cdots m-1)} = 0.$$

Since the second integral is positive and the third integral is negative, α and β can be given *positive* values which will satisfy this condition. The function $\Pi^{(12\cdots m-1)}$ defined by (11) is then positive, continuous on $\mathfrak{X}^{(12\cdots m-1)}$, and is equal to unity except at points of $\mathfrak{P}$ and $\mathfrak{N}$. It is orthogonal to $f_m^{(12\cdots m-1)}$ and hence to all functions of the set $\{f_i^{(12\cdots m-1)}\}$. From the existence of such a function there follows, as we have seen, the existence of a positive continuous function orthogonal to all functions of the given set $\{f_i\}$ and the proof of Theorem I is complete.

8. **Linear integral inequalities.** As an application of the theorem proved in the foregoing sections we consider the following problem.

Given the linear integral inequality

$$\phi(x) + \int_a^b \kappa(x,s)\phi(s)ds > 0 \tag{12}$$

in which the kernel $\kappa(x, s)$ is continuous on the square

$$a \leqq x \leqq b, \qquad a \leqq s \leqq b.$$

Under what conditions will the inequality admit a solution $\phi(x)$ continuous on the interval $\mathfrak{X}$?

We note first that (12) is equivalent to an integral *equation*

$$\phi(x) + \int_a^b \kappa(x,s)\phi(s)ds = \pi(x), \tag{13}$$

where $\pi(x)$ is positive and continuous, but otherwise subject to determination.

If the Fredholm determinant D of the kernel $\kappa(x,s)$ is different from zero, (13) possesses for any continuous $\pi(x)$ a continuous solution

$$\phi(x) = \pi(x) - \frac{1}{D}\int_a^b D(x,s)\pi(s)ds \tag{14}$$

where $D(x, s)$ is the first minor of D. Hence we have the result

If the Fredholm determinant D of the kernel $\kappa(x, s)$ is different from zero, the general solution of the inequality (12) *is given by the formula* (14) *in which $\pi(x)$ is positive and continuous but otherwise arbitrary.*

In case $D=0$, the equation (13) for a given $\pi(x)$ in general admits no solution. A necessary and sufficient condition for the existence of a solution

is that $\pi(x)$ be orthogonal to every solution of the associated homogeneous equation

$$\psi(x) + \int_a^b \psi(s)\kappa(s, x)ds = 0, \tag{15}$$

or what is equivalent, that $\pi(x)$ be orthogonal to every one of a fundamental set of solutions of (15). Suppose such a set is

$$\{\psi_i\} \qquad \psi_1(x), \psi_2(x), \cdots, \psi_m(x).$$

Then the inequality (12) will have a solution if and only if the set of functions $\{\psi_i\}$ admits a positive function $\pi(x)$ orthogonal to all of them.

By our Theorem I, a necessary and sufficient condition for the existence of such a function $\pi(x)$ is that every linear combination of the functions of the set shall change sign. But the linear combinations of these functions constitute the non-trivial* solutions of the equation (15). Hence we have

THEOREM II. *A necessary and sufficient condition that the integral inequality* (12) *admit a solution* $\phi(x)$ *is that every non-trivial solution* $\psi(x)$ *of the associated integral equation* (15) *shall change sign.*†

* By a non-trivial solution we mean a solution which is not identically zero.

† The case in which $D \neq 0$ is compatible with the theorem, since in that case the equation (15) admits no non-trivial solution, and the inequality (12) admits a solution (14).

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A THEOREM ON ORTHOGONAL SEQUENCES*

BY

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1. **Introduction.** In this paper we shall be dealing with infinite sequences of real numbers, and we shall use the functional form of notation. Thus, such a sequence may be considered as a real-valued function $\sigma(i)$ of a variable i, the range of i being the class of positive integers.

A sequence will be said to be *zero* if all its terms are zero, *positive* if all its terms are positive, *negative,* if all its terms are negative, *M-definite* if it has some terms of one sign and no terms of the opposite sign,† *completely signed* if it contains positive terms and negative terms.

If e is any positive number greater than unity, and $\sigma'(i)$ and $\sigma''(i)$ are two sequences such that the two series

$$(1) \qquad \sum_{i=1}^{\infty} |\sigma'(i)|^{e}, \qquad \sum_{i=1}^{\infty} |\sigma''(i)|^{e/(e-1)}$$

converge, then the series

$$(2) \qquad \sum_{i=1}^{\infty}\sigma'(i)\sigma''(i)$$

converges absolutely.‡ If the sum of the series (2) is zero, the sequences σ' and σ'' will be said to be *orthogonal.*

Assuming e to have any *fixed* value greater than unity, let us denote by $\mathfrak{S}'$ the class of all sequences σ' for which the first series of (1) converges, and by $\mathfrak{S}''$ the class of all sequences σ'' for which the second series of (1) converges.

Each of these classes is closed under the operation of linear combination. That is, if $\sigma_1(i), \sigma_2(i), \cdots, \sigma_m(i)$ are sequences of one of these classes, then the sequence $\sigma(i)$ defined by

$$\sigma(i) \equiv \sum_{j=1}^{m} c_j\sigma_j(i),$$

where the c_j are real constants, is a sequence of the same class.

* Presented to the Society, September 8, 1927; received by the editors August 18, 1927.

† An M-definite sequence is an instance of the more general M-definite *function* defined in an earlier paper, *On sets of functions of a general variable*, these Transactions, vol. 29, p. 463.

‡ Cf. F. Riesz, *Les Systèmes d'Equations Linéaires à une Infinité d'Inconnues*, p. 45.

The purpose of the present paper is to prove the

THEOREM.* *A necessary and sufficient condition that there exist in* $\mathfrak{S}''$ *a positive sequence* $\sigma''(i)$ *orthogonal to each of a given set of sequences* $\sigma_1'(i)$, $\sigma_2'(i), \cdots, \sigma_m'(i)$ *in* $\mathfrak{S}'$ *is that no linear combination of the given set shall be M-definite.*

2. **An outline of the proof.** The necessity of the condition is almost obvious. For if $\sigma''(i)$ is orthogonal to each of the given sequences, then it is orthogonal to every linear combination of them; that is

$$\sum_{i=1}^{\infty}\sigma'(i)\sigma''(i) = 0, \tag{3}$$

where $\sigma'(i)$ is any such combination. And the equality (3) is manifestly impossible if $\sigma'(i)$ is M-definite and $\sigma''(i)$ is positive.

To prove the sufficiency of the condition we proceed as follows. By a certain well defined process, the given set of m sequences is replaced by another set of m sequences, called a reduced set. The essential features of the reduction are the following:

(a) One sequence of the reduced set is zero.

(b) If the reduced set admits a positive sequence orthogonal to each of its members, then the same is true of the original set.

The reduction process may be repeated, the property (b) persisting, and the property (a) introducing a new zero sequence at each repetition; so that after m reductions the resulting set contains only sequences which are zero. Since any positive sequence is orthogonal to these zero sequences, it follows from (b) that the given set admits a positive orthogonal sequence. A complication presents itself however in the fact that the sequences of the "reduced sets" are *multiple* sequences, of a type which we now proceed to describe.

3. **Classes of multiple sequences.** A k-tuple sequence may be thought of as a function of k variables, each of which ranges independently over the positive integers, or as a function of a single k-partite variable $q=(i_1, i_2, \cdots, i_k)$ which varies over a composite range $\mathfrak{Q}$ consisting of all k-tuples of positive integers.

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* An analogous theorem relative to continuous functions of a real variable has been proved in an earlier paper, *A theorem on orthogonal functions with an application to integral inequalities*, these Transactions, vol. 30, p. 425. The two theorems justify a certain generalizing postulate which has been used in a theory of linear inequalities in general analysis. Cf. Bulletin of the American Mathematical Society, vol. 33, p. 698.

Any class $\mathfrak{S}$ of simple sequences gives rise to a class of k-tuple sequences which we denote by $\mathfrak{S}_*^k$, and define as follows: The class $\mathfrak{S}_*^k$ consists of all k-tuple sequences $\sigma(i_1, i_2, \cdots, i_k)$ for which there exist sequences $\sigma_1(i_1), \sigma_2(i_2), \cdots, \sigma_k(i_k)$ of $\mathfrak{S}$ such that

$$| \sigma(i_1, i_2, \cdots, i_k) | \leqq \sigma_1(i_1)\sigma_2(i_2) \cdots \sigma_k(i_k).$$

Thus the classes $\mathfrak{S}'$ and $\mathfrak{S}''$ defined in §1 give rise to classes of multiple sequences $\mathfrak{S}_*'^k$ and $\mathfrak{S}_*''^k$. Relative to these two classes we note the following properties:

(i) Each of the classes $\mathfrak{S}_*'^k$ and $\mathfrak{S}_*''^k$ is closed under the process of linear combination.

(ii) If $\sigma'(i_1, i_2, \cdots, i_k)$ and $\sigma''(i_1, i_2, \cdots, i_k)$ are sequences of $\mathfrak{S}_*'^k$ and $\mathfrak{S}_*''^k$ respectively, then the multiple series

$$\sum_{i_1 i_2 \cdots i_k} \sigma'(i_1, i_2, \cdots, i_k)\sigma''(i_1, i_2, \cdots, i_k) \tag{4}$$

converges absolutely.

(iii) If $\sigma'(i_1, i_2, \cdots, i_k)$ belongs to $\mathfrak{S}_*'^k$ and $\sigma''(i_1, i_2, \cdots, i_l)$ belongs to $\mathfrak{S}_*''^l$ where $l = k + h$, then the series

$$\sum_{i_1 i_2 \cdots i_k} \sigma'(i_1, i_2, \cdots, i_k)\sigma''(i_1, i_2, \cdots, i_l)$$

converges absolutely for every $(i_{k+1}, i_{k+2}, \cdots, i_l)$, and the resulting sum is a sequence $\sigma''(i_{k+1}, i_{k+2}, \cdots, i_l)$ of the class $\mathfrak{S}_*''^h$.

Definition. If the sum of the series (4) is zero, then the k-tuple sequences σ' and σ'' will be said to be *orthogonal*.

4. **Reduction.** Consider any k-tuple* sequence $\rho(q) = \rho(i_1, i_2, \cdots, i_k)$ which is completely signed (that is, which contains at least one positive and one negative term), the symbol ρ being suggestive of the special rôle of *reducing sequence*.

Relative to the sequence $\rho(q)$, the range $\mathfrak{Q}$ of the variable q can be divided into three well defined sub-classes:

$$\mathfrak{Q}_P^{(\rho)} \equiv [\text{all } q \text{ such that } \rho(q) > 0],$$

$$\mathfrak{Q}_Z^{(\rho)} \equiv [\text{all } q \text{ such that } \rho(q) = 0],$$

$$\mathfrak{Q}_N^{(\rho)} \equiv [\text{all } q \text{ such that } \rho(q) < 0].$$

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* A simple sequence if $k = 1$.

The three classes are mutually exclusive, and are complementary, that is, $\mathfrak{Q} \equiv \mathfrak{Q}_{P^{(\rho)}} + \mathfrak{Q}_{Z^{(\rho)}} + \mathfrak{Q}_{N^{(\rho)}}$. The elements of the respective classes will be denoted by the appropriate small letters:

$$\mathfrak{Q}_{P^{(\rho)}} \equiv [p], \quad \mathfrak{Q}_{Z^{(\rho)}} \equiv [z], \quad \mathfrak{Q}_{N^{(\rho)}} \equiv [n].$$

Corresponding to the division of the range $\mathfrak{Q}$, any k-tuple sequence $\sigma(q)$ can be divided into three well defined *sections*: $\sigma(p)$, $\sigma(z)$, $\sigma(n)$; and it will be convenient to use the notation

$$\sigma(q) \equiv [\sigma(p), \sigma(z), \sigma(n)]$$

to bring the sections into evidence. The reducing sequence, which for simplicity is omitted from the notation, will always be known from the context or by explicit statement.

In the sequel it will sometimes be desirable to replace one or more of the sections of a sequence by the corresponding sections of the identically zero sequence $\omega(q)$:

$$\omega(q) \equiv [\omega(p), \omega(z), \omega(n)] \equiv 0.$$

Of particular importance are the *reduced sequences* of the following two special types:

$$\sigma_{PZ}(q) \equiv [\sigma(p), \sigma(z), \omega(n)], \quad \sigma_N(q) \equiv [\omega(p), \omega(z), \sigma(n)]. \tag{5}$$

We note the obvious property that if $\sigma(q)$ belongs to the class $\mathfrak{S}_*'^k$, then each of the reduced sequences (5) belongs to this class.

5. **The reduced outer product.** The reducing k-tuple sequence $\rho(q)$ determines with any second k-tuple sequence $\sigma(q)$ a certain $2k$-tuple sequence called their *reduced outer product*, denoted by $((\rho\sigma))$, and defined as follows:

$$((\rho\sigma)) \equiv \rho_{PZ}(q_1)\sigma_P(q_2) - \sigma_{PZ}(q_1)\rho_N(q_2), \tag{6}$$

where the variables q_1 and q_2 vary independently over the range $\mathfrak{Q}$ and the reductions are made with respect to the first factor $\rho(q)$ as reducing sequence.

This type of combination of sequences is clearly not in general commutative. It does possess the following noteworthy properties:

(i) If σ is zero, or if $\sigma = \rho$, then $((\rho\sigma))$ is zero.

(ii) If ρ and σ belong to $\mathfrak{S}_*'^k$, then $((\rho\sigma))$ belongs to $\mathfrak{S}_*'^{2k}$.

6. **Reduction of a set of sequences.** Suppose we have a set of m k-tuple sequences

$$\sigma_1'(q), \quad \sigma_2'(q), \cdots, \sigma_m'(q), \tag{7}$$

belonging to $\mathfrak{S}_*'^k$. Then relative to any one of them which is completely

signed, say $\sigma_r'(q)$, we may form a *reduced set*, viz. a set of m $2k$-tuple sequences

$$((\sigma_r'\sigma_1')),\ ((\sigma_r'\sigma_2')),\ \cdots,\ ((\sigma_r'\sigma_m')), \tag{8}$$

each of which is the reduced outer product of the corresponding sequence of the given set by $\sigma_r'(q)$.

We shall make use of the following three properties of this reduction process:

PROPERTY (a). *The rth sequence of the reduced set is zero.*

PROPERTY (b). *If there is in $\mathfrak{S}_*''^{2k}$ a positive 2k-tuple sequence which is orthogonal to each sequence of the reduced set* (8), *then there is in $\mathfrak{S}_*''^{k}$ a positive k-tuple sequence which is orthogonal to each sequence of the set* (7).

PROPERTY (c). *If the set* (7) *admits no M-definite linear combination, then the same is true of the reduced set* (8).

Property (a) is an immediate consequence of §5 (i).

To prove property (b), we have by hypothesis a positive $2k$-tuple sequence $\sigma''(q_1, q_2)$ such that for $j=1, 2, \cdots, m$,

$$\sum_{q_1q_2}[\sigma_{rPZ}'(q_1)\sigma_{jN}'(q_2) - \sigma_{jPZ}'(q_1)\sigma_{rN}'(q_2)]\sigma''(q_1,q_2) = 0.$$

Since the series on the left converges absolutely, we may write this in the form

$$\sum_{q_2}\left[\sum_{q_1}\sigma_{rPZ}'(q_1)\sigma''(q_1,q_2)\right]\sigma_{jN}'(q_2) - \sum_{q_1}\left[\sum_{q_2}\sigma_{rN}'(q_2)\sigma''(q_1,q_2)\right]\sigma_{jPZ}'(q_1)=0. \tag{9}$$

From §3 (iii), it follows that each of the expressions in square brackets is a sequence of $\mathfrak{S}_*''^{k}$, and it is clear that the first of these sequences is positive and the second negative. If we denote them for the moment by $\pi''(q)$ and $\nu''(q)$ respectively, we may write (9) in the form

$$\sum_{q}\sigma_{jN}'(q)\pi''(q) - \sum_{q}\sigma_{jPZ}'(q)\nu''(q) = 0. \tag{10}$$

Now, defining the k-tuple sequence $\sigma''(q)$ by the equality

$$\sigma''(q) = \pi_N''(q) - \nu_{PZ}''(q)$$

the reductions being relative to the reducing sequence $\sigma_r'(q)$, we replace (10) by the equivalent equation

$$\sum_q \sigma_j'(q)\sigma''(q) = 0. \tag{11}$$

The sequence $\sigma''(q)$ is a *positive* sequence of $\mathfrak{S}_*''^k$, and since the relation (11) holds for $j=1, 2, \cdots, m$, we have established property (b).

To prove property (c) indirectly, let us suppose there exist constants $c_1, c_2, \cdots, c_m$, such that*

$$\sum_{j=1}^{m} c_j((\sigma_r' \sigma_j')) \geqq{}' 0. \tag{12}$$

Since $((\sigma_r' \sigma_r'))$ is zero, the coefficient c_r is arbitrary, and the term corresponding to $j=r$ may be omitted from the summation. Denoting this omission by an apostrophe over the summation sign, and substituting their definitional values for $((\sigma_r' \sigma_j'))$, we write (12) in the form

$$\sum_{j=1}^{m}{}' c_j\{\sigma_{rPZ}'(q_1)\sigma_{jN}'(q_2) - \sigma_{jPZ}'(q_1)\sigma_{rN}'(q_2)\} \geqq{}' 0.$$

And to bring the sections of the reduced sequences into evidence we write in the more extended form

$$\begin{aligned}\sum_{j=1}^{m}{}' c_j\{ &[\sigma_r'(p_1), \sigma_r'(z_1), \omega(n_1)][\omega(p_2), \omega(z_2), \sigma_j'(n_2)] \\ &- [\sigma_j'(p_1), \sigma_j'(z_1), \omega(n_1)][\omega(p_2), \omega(z_2), \sigma_r'(n_2)]\} \geqq{}' 0.\end{aligned} \tag{13}$$

Recalling that the ranges of the variables p_1, z_1, n_1, p_2, z_2, n_2 do not overlap, and that the sections $\sigma_r'(z)$, $\omega(p)$, $\omega(z)$, $\omega(n)$ are identically zero, we may replace (13) by two simpler simultaneous inequalities

$$\sum_{j=1}^{m}{}' c_j\{\sigma_r'(p_1)\sigma_j'(n_2) - \sigma_j'(p_1)\sigma_r'(n_2)\} \geqq 0, \tag{14}$$

$$- \sum_{j=1}^{m}{}' c_j\sigma_j'(z_1)\sigma_r'(n_2) \geqq 0, \tag{15}$$

the symbol $\geqq$ having the significance of $\geqq'$ in at least one of them.

Now since by definition the product $-\sigma_r'(p_1)\,\sigma_r'(n_2)$ is positive for every (p_1, n_2), we may obtain from (14) an equivalent inequality

$$\sum_{j=1}^{m}{}' c_j \frac{\sigma_j'(p_1)}{\sigma_r'(p_1)} \geqq \sum_{j=1}^{m}{}' c_j \frac{\sigma_j'(n_2)}{\sigma_r'(n_2)}. \tag{16}$$

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* The symbol $\geqq'$ is to be read "is somewhere greater than and nowhere less than." Thus (12) is equivalent to the statement that the left side is M-definite.

The values on the left side of (16) have (for varying p_1) a greatest lower bound, and those on the right (for varying n_2) a least upper bound. These bounds may or may not coincide, but in any case we may choose the arbitrary c_r so that

$$\sum_{j=1}^{m}{}' c_j \frac{\sigma_j'(p_1)}{\sigma_r'(p_1)} \geqq -c_r \geqq \sum_{j=1}^{m}{}' c_j \frac{\sigma_j'(n_2)}{\sigma_r'(n_2)} \cdot$$

From this double relation we obtain, since $\sigma_r'(p)>0$ and $\sigma_r'(n)<0$,

$$(17) \qquad \sum_{j=1}^{m} c_j\sigma_j'(p_1) \geqq 0, \qquad \sum_{j=1}^{m} c_j\sigma_j'(n_2) \geqq 0.$$

Furthermore, from (15) we obtain

$$(18) \qquad \sum_{j=1}^{m} c_j\sigma_j'(z_1) \geqq 0.$$

Since the ranges of p_1, z_1, n_2, when combined adjunctively, form the range $\mathfrak{Q}$, we may combine (17) and (18) in the single inequality

$$\sum_{j=1}^{m} c_j\sigma_j'(q) \geqq' 0.$$

This result contradicts the hypothesis of property (c), and the contradiction proves the desired conclusion.

7. **Completion of proof of the theorem.** We return now to the principal theorem of §1. By hypothesis we are given a set of sequences

$$\sigma_1'(i),\ \sigma_2'(i),\ \cdots,\ \sigma_m'(i),$$

of $\mathfrak{S}'$, of which no linear combination is M-definite. We are to prove that there is a positive sequence of $\mathfrak{S}''$ which is orthogonal to each sequence of the set.

First of all we may assume that not all the sequences of the set are zero, since in that case the theorem is obviously true. Also, the sequences which are not zero are completely signed, since they cannot be M-definite.

Suppose the first sequence σ_1' is not zero. We form the reduced set with respect to σ_1' as reducing function, and denote it by

$$0,\ \sigma_2'^{(1)},\ \sigma_3'^{(1)},\ \cdots,\ \sigma_m'^{(1)}.$$

It consists of m double sequences of $\mathfrak{S}_*'^2$, of which the first is zero. Other sequences of the set may be zero. If they are all zero, they all admit a positive orthogonal sequence, from which follows our theorem, by property

(b). In any case, the non-zero sequences are completely signed, by property (c). We choose the first such sequence—suppose for definiteness it is $\sigma_2'^{(1)}$—as a reducing sequence, and form a second reduced set

$$0,\ 0,\ \sigma_3'^{(2)},\ \cdots,\ \sigma_m'^{(2)}.$$

The process is now obvious. We repeat the reduction process, until after $l(\leqq m)$ reductions we obtain a set of h-tuple sequences ($h=2^l$), all of which are zero.

Any positive sequence of $\mathfrak{S}_*''^h$ is orthogonal to all sequences of this last reduced set, and the existence of such a positive sequence implies, by repeated application of property (b), the existence of a positive sequence in $\mathfrak{S}''$ orthogonal to each sequence of the given set. This completes the proof of the theorem.

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